Two-regular subgraphs of hypergraphs

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Abstract

We prove that the maximum number of edges in a k-uniform hypergraph on n vertices containing no two-regular subhypergraph is $\binom{n-1}{k-1}$ if $k \geq 4$ is even and n is sufficiently large. Equality holds only if all edges contain a specific vertex v. For odd k we conjecture that this maximum is $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$, with equality only for the hypergraph described above plus a maximum matching omitting v.

1 Introduction

One of the most basic facts in combinatorics is that an acyclic graph on n vertices has at most $n-1$ edges, with equality only for trees. A natural generalization to hypergraphs (see Berge [3] for more details) is obtained by defining a circuit to be a hypergraph consisting of distinct vertices $v_1, v_2, \dots, v_k$ and distinct edges $e_1, \dots, e_k$ such that $v_i \in e_i$ for $i = 1, 2 \dots, k$, $v_{i+1} \in e_i$ for $i = 1, 2 \dots, k-1$, and $v_1 \in e_k$. Then a hypergraph H with no circuit satisfies

$$\sum_{e \in H} (|e| - 1) \leq |V(H)| - 1.$$

In this paper, we consider a generalization to hypergraphs in a different direction. Since a cycle is a two-regular graph, we may ask for the maximum number of edges that a hypergraph on n vertices can have without a two-regular subgraph – i.e. a subhypergraph in which every vertex has degree two. Throughout the paper, hypergraphs where all edges have size k are called k-uniform hypergraphs or, simply, k-graphs. Let $\mathcal{P}_k(n)$ be the family of n-vertex k-graphs containing no two-regular subgraph. Fixing k, a star hypergraph consists of all k-element sets of vertices containing a fixed vertex. Our main result shows that star hypergraphs are extremal in $\mathcal{P}_k(n)$ when k is even:

Theorem 1. *For every even integer $k > 2$, there exists an integer n_k such that for $n \geq n_k$, if $H \in \mathcal{P}_k(n)$ then $|H| \leq \binom{n-1}{k-1}$. Equality holds if and only if H is a star.*

The non-uniform analog of this theorem, which is much simpler, is proved in Section 2. As one might expect, the proof of Theorem 1 needs completely new techniques than the graph case.

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The result is proved via the stability approach. Stability results were introduced in extremal graph theory by Erdős and Simonovits [14] in the 60's. The program of using stability to prove exact results has been recently used with great success in extremal set theory (see [5, 6, 7, 8, 9, 10, 11]). The specific approach in this paper has similarities to [10]. Perhaps the main difficulty in passing to an exact result when k is odd is that stars are not extremal in $\mathcal{P}_k(n)$ when k is odd: it is possible to add to a star on n vertices a matching of size $\lfloor \frac{n-1}{k} \rfloor$, resulting in a k-graph in $\mathcal{P}_k(n)$ with a few more edges. We conjecture that this "star-plus-matching" construction is the unique extremal configuration when k is odd:

Conjecture 1. *For every odd integer $k \geq 3$, there exists an integer n_k such that for $n \geq n_k$, if $H \in \mathcal{P}_k(n)$ then $|H| \leq \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$. Equality holds if and only if H is a star with center v together with a maximal matching omitting v.*

Conjecture 1 is a weaker version of a conjecture due to Füredi, that for $k > 3$, a k-graph containing no two pairs of disjoint sets with the same union has at most $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$ edges. For odd $k > 3$, this implies Conjecture 1; in fact a pair of disjoint sets with the same union is the smallest possible two-regular k-graph when k is odd. The question of determining the maximum number of edges $f_k(n)$ of a k-graph on n vertices containing no two pairs of disjoint edges with the same union was originally raised by Erdős (see [4]). This problem was studied by Frankl and Füredi [4], and the authors [12], who showed that $f_k(n) < 3\binom{n}{k-1}$, and this is the current best upper bound on $f_k(n)$.

This paper is organized as follows. In the next section, we prove the nonuniform analogue of Theorem 1, that a collection of subsets of an n-element set with no 2-regular subsystem has size at most 2^{n-1} with equality (for $n \geq 3$) only for a star. In Section 3, we present two lemmas used to prove Theorem 1. The proof of Theorem 1 is in Sections 4–6, and has three parts. First we shall show (see Section 4) that if $H \in \mathcal{P}_k(n)$ then $|H| \lesssim \binom{n-1}{k-1}$. Using this result, we prove the stability result (see Section 5), which says that if $|H| \sim \binom{n-1}{k-1}$ then $\triangle(H) \sim \binom{n-1}{k-1}$. Finally, we use this stability theorem to prove Theorem 1 in Section 6. The final section mentions related open problems.

Terminology. We denote by $V(H)$ the set of vertices of a hypergraph H. The degree of a vertex v, written $d(v)$, is the number of edges containing that vertex. A matching is a hypergraph whose edges are pairwise disjoint – equivalently this is a hypergraph in which every vertex has degree one. A k-graph is a hypergraph where all sets have size k, and a hypergraph is r-regular if all its vertices have degree r. We write $\binom{X}{k}$ for the collection of all k-sets of X. A star is a hypergraph or a k-graph consisting of all possible edges containing a fixed vertex. For a hypergraph H, denote by $\triangle(H)$ its maximum degree and $d(H)$ its average degree. For $v \in V(H)$, let $H - \{v\} = \{e \in H : v \not\in e\}$ and $H_v = \{e \setminus \{v\} : v \in e \in H\}$. If $f, g : \mathbb{N} \to \mathbb{R}$ are two functions then we write $f(n) \gtrsim g(n)$ to denote that $f(n) \geq g(n)h(n)$ for some function $h(n)$ such that $\liminf_{n\to\infty} h(n) = 1$. This is an equivalent but more convenient way to write $f(n) \geq (1 + o(1))g(n)$. In the case $f(n) = (1 + o(1))g(n)$ we write $f(n) \sim g(n)$. If there is a constant $c > 0$ such that $f(n) \geq cg(n)$ for all n, then we write $f(n) \gg g(n)$. Throughout this paper, all asymptotic statements are taken as $n \to \infty$, and k is always fixed relative to n.

2 Non-uniform hypergraphs

In this section, we observe the following simple theorem. Let $\mathcal{P}(n)$ denote the family of hypergraphs (without restriction on the sizes of the edges) on n vertices containing no two-regular subgraph. We stipulate that edges of a hypergraph are non-empty sets. A star on n vertices is a hypergraph consisting of all 2^{n-1} sets containing a fixed vertex.

Theorem 2. *Let $n \geq 1$ and $H \in \mathcal{P}(n)$. Then $|H| \leq 2^{n-1}$. If $n \geq 3$ and equality holds, then H is a star.*

Proof. We remark that it is easy to obtain an upper bound 2^{n-1}: if $H \in \mathcal{P}(n)$, then H contains at most one complementary pair – a complementary pair consists of the edge e and the edge $V(H)\backslash\{e\}$. This shows $|H| \leq 2^{n-1}+1$, but if H contains both edges of some complementary pair, then $V(H)$ cannot be an edge of H, showing $|H| \leq 2^{n-1}$. For the characterization of equality, we proceed by induction on n for $n \geq 3$.

It is straightforward to check the case $n = 3$; we omit the details. Now we proceed to the induction step. Let us assume that $n \geq 4$ and $H \in \mathcal{P}(n)$ has size $|H| = 2^{n-1}$. We will show that H is a star. Since a star is a maximal two-regular subgraph, this proves Theorem 2. First we show that every vertex of H has degree exactly 2^{n-2}. If there is a vertex $v \in V(H)$ with $d(v) < 2^{n-2}$, then $H - \{v\}$ has a two-regular subgraph, by induction. So every vertex of H has degree at least 2^{n-2}. Pick a vertex $x \in V(H)$. If all sets contain x then we're done, so we may assume that there is a set $e \in H$ of size k missing x. For each subset $f \subset V(H)\backslash(e \cup \{x\})$, the number of sets in H containing x whose intersection with $V(H)\backslash(e \cup \{x\})$ is f is at most 2^{k-1}, for otherwise two of these sets have complementary intersections in e and these together with e give a two-regular subgraph, contradicting $H \in \mathcal{P}(n)$. Hence the number of sets containing x is at most $2^{n-k-1}2^{k-1} = 2^{n-2}$. So every vertex of H has degree exactly 2^{n-2}. In particular, for any $x \in V(H)$, $|H - \{x\}| = 2^{n-2}$ so by induction, $H - \{x\}$ is a star with center at some vertex w. But now it is easy to see that all sets containing x must also contain w, else we find a two-regular subgraph (either x lies in 2 sets omitting w, or one set omitting w and one containing w since $2^{n-2} > 1$; in each case choose an appropriate set containing w to form a two-regular subgraph). Therefore H is a star with center w. □

3 Preliminary Lemmas

In this section, we present two lemmas which will be used in proving Theorem 1. The first lemma involves matchings. If M_1 and M_2 are distinct matchings and $V(M_1) = V(M_2)$, then $M_1 \triangle M_2$ is a hypergraph whose vertices all have degree two. This observation is the key point of the following lemma.

Lemma 1. *Let $H \in \mathcal{P}_k(n)$ be a k-graph with $d(H) = d$ and $\triangle(H) = \triangle$. Then $\triangle \geq \frac{1}{k}\left(\frac{d}{e}\right)^{\frac{k}{k-1}}$. In particular, $d \leq ek\Delta^{(k-1)/k}$ and so $|H| \leq en\Delta^{(k-1)/k}$.*

Proof. Suppose $H \in \mathcal{P}_k(n)$ is a k-graph of average degree d such that $\triangle(H) = \triangle$ is less than the bound in the theorem. We count matchings in H of size $m = \lfloor |E|/k\triangle \rfloor$ to prove

$H \notin \mathcal{P}_k(n)$. For a lower bound on the number of matchings of size m, we may greedily pick disjoint edges $e_1, e_2, \ldots, e_m$ where at each step we exclude all edges that intersect previously chosen edges. Since at each step we exclude at most $k\Delta$ new edges, the number of matchings of size m in H is at least

$$\frac{1}{m!}\prod_{i=1}^{m}(|E| - k\Delta i) = \frac{1}{m!}|E|^m \prod_{i=1}^{m}\left(1 - \frac{k\Delta i}{|E|}\right) = \frac{1}{m!}|E|^m \prod_{i=1}^{m-1}\left(1 - \frac{i}{m}\right) > (k\Delta)^m.$$

To complete the proof, we show that there exist distinct matchings M_1, M_2 of H such that $\bigcup_{e\in M_1} e = \bigcup_{e \in M_2} e$. This suffices, since the edges in $M_1 \triangle M_2$ form a two-regular subgraph, thus contradicting $H \in \mathcal{P}_k(n)$. Using the upper bound on $\triangle$:

$$\binom{n}{mk} < \left(\frac{\mathrm{e}n}{mk}\right)^{mk} \le \left(\frac{\mathrm{e}k\triangle}{d}\right)^{km} < (k\triangle)^m.$$

Since $\binom{n}{mk}$ is the number of sets of mk vertices of H, and there are more than $(k\triangle)^m$ matchings of size m in H, we find the two required distinct matchings M_1, M_2. □

Our second lemma involves circuits in hypergraphs. A circuit is a hypergraph consisting of distinct vertices $v_1, v_2, \ldots, v_k$ and distinct edges $e_1, \ldots, e_k$ such that $v_i \in e_i$ for $i = 1, 2 \ldots, k$, $v_{i+1} \in e_i$ for $i = 1, 2 \ldots, k-1$, and $v_1 \in e_k$. We require the following lemma on circuits in hypergraphs of a certain bipartite structure:

Lemma 2. *Let G be a k-graph and $V(G) = A \cup B$, where $A \cap B = \emptyset$, all edges $e \in G$ have $|e \cap A| = k-1$, and every $(k-1)$-set in A lies in at least two edges of G. If $G \in \mathcal{P}_k(n)$, then*

$$|G| < 2|B|\binom{|A|+k-3}{k-2}.$$

Proof. It is enough to show $|G| < 2|B|\binom{|A|-1}{k-2}$ when $|A| = a(k-1)$ for some integer a, since we may always add at most $k-2$ points to A so that $k-1$ divides $|A|$. Baranyai's Theorem [2] states that if s divides n, then the complete s-graph on n vertices can be partitioned into $\binom{n-1}{s-1}$ perfect matchings. Using this theorem with $s = k-1$, we write

$$\binom{A}{k-1} = M^1 \cup M^2 \cup \cdots \cup M^{\binom{|A|-1}{k-2}}$$

where each M^i is a matching and the matchings are edge-disjoint. For each matching $M^i = \{e_1^i, \ldots, e_a^i\}$, let G^i be the set of edges in E whose intersection with A is e_j^i for some j. Let f_j^i be the set of vertices $v \in B$ such that $e_j^i \cup \{v\} \in G^i$. Consider the hypergraph H_j^i with edges f_j^i, $j = 1, \ldots, a$. If H_j^i contains a circuit with vertices $v_1, v_2, \ldots, v_p$, then G contains the two-regular subgraph with edges

$$e_1^i \cup \{v_1\} \quad e_1^i \cup \{v_2\} \quad e_2^i \cup \{v_2\} \quad e_2^i \cup \{v_3\} \quad e_p^i \cup \{v_p\} \quad e_p^i \cup \{v_1\}$$

which contradicts $G \in \mathcal{P}_k(n)$. Consequently, H_j^i has no circuit. It is well-known that a hypergraph H with no circuit satisfies

$$\sum_{e \in H}(|e| - 1) \le (|V(H)| - 1). \tag{5}$$

By hypothesis, $|f_j^i| \geq 2$ for all i, j. Applying (5) to H_j^i, we therefore obtain

$$\sum_j |f_j^i| \leq \sum_j 2(|f_j^i| - 1) < 2|B|. \tag{6}$$

Adding (6) over different i, j, we obtain

$$|G| \leq \sum_i \sum_j |f_j^i| \leq 2|B|\binom{|A|-1}{k-2}. \quad \square$$

4 The Asymptotic Result

Theorem 3. *Let $k \geq 3$ and $H \in \mathcal{P}_k(n)$. Then $|H| - \binom{n-1}{k-1} \ll n^{k-1-1/11}$.*

Proof. We prove the following more precise statement: for all $n > k^{100}$,

$$|H| < (1 + cn^{-\gamma})\binom{n-1}{k-1}$$

where $c = 2(k+1)!$ and $\gamma = \frac{1}{11}$. Define $\alpha = (k+1)/(3k-1)$ for $k > 3$ and $\alpha = 7/11$ for $k = 3$. Suppose, for a contradiction, that $|H|$ equals this upper bound (rounded up if necessary) for some $H \in \mathcal{P}_k(n)$. Let T denote the set of vertices of H of degree at least $\triangle = n^{k-1-\alpha}$, and set $t = |T|$. Then $t\triangle \leq k|H|$ and, since $n > k^{100}$,

$$t < \triangle^{-1} k(1 + cn^{-\gamma})\binom{n-1}{k-1} < kn^{\alpha}. \tag{9}$$

Let $H_i = \{e \in H : |e \cap T| = i\}$ for $i \leq k$, and define $G = \{e \in H_1 : \nexists f \in H_1 : e \setminus T = f \setminus T\}$. In particular, $|G| \leq \binom{n}{k-1}$.

Claim 1. $|H_i| < \begin{cases} en^{1+(k-1)(k-1-\alpha)/k} & \text{for } i = 0 \\ |G| + 2kn^{k-2+\alpha} & \text{for } i = 1 \end{cases}$

Proof. Since $\triangle(H_0) < \triangle$, by definition of T, the first bound follows from Lemma 1. For the second bound, we apply Lemma 2 to $H_1 \backslash G$ with $A = V(H) \backslash T$ and $B = T$ to obtain $|H_1 \backslash G| < 2|T|\binom{n+k-3}{k-2} < 2tn^{k-2}$. The bound on $|H_1|$ now follows from (9).

Claim 2. $|H \backslash (H_0 \cup H_1)| < \begin{cases} k^2 n^{k-2+2\alpha} & \text{for } k > 3 \\ 6(n^{1+\alpha} + n^{3\alpha}) & \text{for } k = 3 \end{cases}$

Proof. For $k > 3$, by definition, every edge in $H \backslash (H_0 \cup H_1)$ consists of two vertices of T and $k - 2$ vertices of $V(H)$, so certainly $|H \backslash (H_0 \cup H_1)| \leq \binom{|T|}{2} n^{k-2}$. Now apply (9). For $k = 3$, observe that $|H_3| < \binom{|T|}{3}$. Furthermore, by Lemma 2, with $A = T$ and $B = V(H) \backslash T$, $|H_2| < 2|T|(n - |T|) + \binom{|T|}{2} < 2tn$. Using (9) gives the claim.

Now we complete the proof. By definition of α, the bounds in Claims 1 and 2 are all of order at most $n^{k-1-\gamma}$ (the case $i = 0$ in Claim 1 needs a somewhat tedious calculation). Specifically,

$$|H\backslash G| = |H_0| + |H_1\backslash G| + |H\backslash(H_0 \cup H_1)| < (k^2 + 2k + e)n^{k-1-\gamma} < 2k^2 n^{k-1-\gamma}. \tag{10}$$

Using the bound $|G| \le \binom{n}{k-1}$ in (10), we obtain

$$|H| = |G| + |H\backslash G| < \binom{n}{k-1} + 2k^2 n^{k-1-\gamma} < (1 + cn^{-\gamma})\binom{n-1}{k-1}.$$

This contradiction completes the proof. □

5 Stability

Theorem 4. *Let $k \ge 3$ and $H_n \in \mathcal{P}_k(n)$. If $|H_n| \sim \binom{n-1}{k-1}$, then $\triangle(H_n) \sim \binom{n-1}{k-1}$.*

Proof. For simplicity of notation, we let $H = H_n$ and omit the subscript n when dealing with hypergraphs constructed from H. As in the proof of Theorem 3, let T denote the set of vertices in H of degree at least $n^{k-1-\alpha}$, $H_1 = \{e \in H : |e \cap T| = 1\}$ and $G = \{e \in H_1 : \nexists f \in H_1 : e \setminus T = f \setminus T\}$. Define

$$G' = \{e \setminus T : e \in G\}.$$

For each $x \in T$, let $G_x = \{e \in G' : e \cup \{x\} \in G\}$. We assume that $|G_v| = \max_{x \in T} |G_x|$. Note that all sets in G have size k, and all sets in G' or any G_x have size $k - 1$. By (10), $|G'| = |G| \sim |H| \sim \binom{n-1}{k-1}$, so it suffices to prove $|G_v| \sim |G'|$ to prove the theorem. Suppose, for a contradiction, that for some positive $\varepsilon < \frac{1}{2}$,

$$|G_v| \lesssim (1 - \varepsilon)|G'|. \tag{12}$$

Let $P(G') = \{\{e, f\} \subset G' : |e \cap f| = 1\}$. Define $P_1(G') \subset P(G')$ to be the set of pairs $\{e, f\} \in P(G')$ such that $e, f \in G_x$ for some x, and $P_2(G') = P(G')\backslash P_1(G')$. The strategy is to use (12) to derive a contradiction by finding edges $e, e' \in G_x$ and $f, f' \in G_y$, for some $x \ne y$, such that $|e \cap f| = 1 = |e' \cap f'|$, $e \triangle f = e' \triangle f'$ and $e \cap f \ne e' \cap f'$ (sometimes the latter condition will be guaranteed by $e \cap e' = \emptyset = f \cap f'$). For in this case, the edges

$$e \cup \{x\} \quad e' \cup \{x\} \quad f \cup \{y\} \quad f' \cup \{y\} \tag{13}$$

form a two-regular subgraph of H.

Claim 1. $|P_2(G')| \le \frac{1}{2}\binom{t}{2}\binom{2k-4}{k-2}\binom{n-1}{2k-4}$.

Proof. Fix distinct $x, y \in T$ and let us first consider distinct $\{e_i, f_i\} \in P_2(G')$ such that $e_i \in G_x$ and $f_i \in G_y$. Suppose that $e_i \triangle f_i = e_j \triangle f_j$ for all $1 \le i < j \le \frac{1}{2}\binom{2k-4}{k-2} + 1$. If $e_i \cap f_i \ne e_j \cap f_j$ for some $i \ne j$, then we obtain a two-regular subgraph as in (13). So we may assume that $e_i \cap f_i = e_j \cap f_j$ for all $i \ne j$. Since we also have $e_i \triangle f_i = e_j \triangle f_j$, and the number of pairs $\{e_i, f_i\}$ is more than $\frac{1}{2}\binom{2k-4}{k-2}$, this implies that $\{e_i, f_i\} = \{e_j, f_j\}$ for some $i \ne j$, contradiction. The argument above shows that each $(2k-4)$-set occurs as $e \triangle f$ at most $\frac{1}{2}\binom{2k-4}{k-2}$ times. Consequently, the number of $\{e, f\} \in P_2(G')$ such that $e \in G_x$ and $f \in G_y$ is at most $\frac{1}{2}\binom{2k-4}{k-2}\binom{n-1}{2k-4}$. It follows that $|P_2(G')| \le \frac{1}{2}\binom{t}{2}\binom{2k-4}{k-2}\binom{n-1}{2k-4}$.

For the rest of the proof, let $\psi(\varepsilon) = ((1-\varepsilon)^2+\varepsilon^2)^{1/2}$. For $i \in \{1,2\}$, let $Q_i(G')$ denote the set of pairs $\{\{e,f\},\{e',f'\}\}$ such that $\{e,f\},\{e',f'\} \in P_i(G')$, $e \cap e' = \emptyset = f \cap f'$ and $e \triangle f = e' \triangle f'$. These are called type i quadrilaterals of G'. For $x \in T$, define $Q_1(G_x)$ to be the collection of pairs $\{\{e,f\},\{e',f'\}\} \in Q_1(G')$ such that $\{e,f,e',f'\} \subset G_x$. These are type 1 quadrilaterals of G_x. Let K be the complete $(k-1)$-graph on $V(G')$.

Claim 2. $|P_1(G')| \lesssim \psi(\varepsilon)\cdot|P(K)|$.

Proof. Let $\{\{e,f\},\{e',f'\}\} \in Q_1(G')$. If $e,f \in G_x$ and $e',f' \in G_y$ with $x \neq y$, then we obtain a two-regular subgraph similar to that in (13). We conclude that if $e,f \in G_x$, then also $e',f' \in G_x$. It follows that

$$|Q_1(G')| = \sum_{x\in T} |Q_1(G_x)|. \tag{14}$$

For a pair $\{g,h\}$ of disjoint sets of size $k-2$ in $V(G')$, let $p_1(g,h)$ denote the number of pairs $\{e,f\} \in P_1(G')$ with $e\backslash f = g$ and $f\backslash e = h$. The number of such pairs $\{g,h\}$ is at most

$$\binom{\binom{n-1}{k-2}}{2} := N.$$

Note also that the sum of $p_1(g,h)$ over all $\{g,h\} \subset V(G')$ is exactly $|P_1(G')|$. By convexity of binomial coefficients,

$$|Q_1(G')| \;=\; \sum_{\{g,h\}} \binom{p_1(g,h)}{2} \;\gtrsim\; \binom{|P_1(G')|/N}{2}\cdot N \;\sim\; \frac{|P_1(G')|^2}{\binom{n-1}{k-2}^2}. \tag{15}$$

On the other hand, we observe that $|Q_1(G_x)| \leq \frac{1}{2}(k-1)^2\binom{|G_x|}{2}$, since if we fix two disjoint edges, say $e,e' \in G_x$, then the number of type 1 quadrilaterals of the form $\{\{e,f\},\{e',f'\}\}$ is at most $(k-1)^2$. The same type 1 quadrilaterals are counted if we had fixed the two disjoint edges $f,f' \in G_x$ instead of e,e', and this gives the observation. Therefore, by (14),

$$|Q_1(G')| \leq \frac{1}{2}(k-1)^2 \sum_{x\in T}\binom{|G_x|}{2}.$$

This sum is a maximum when $|G_v| \sim (1-\varepsilon)|G'|$ and $|G_w| \sim \varepsilon|G'|$ for some $w \neq v$, and the rest of the $|G_x|$s are zero. Therefore

$$|Q_1(G')| \leq \frac{1}{4}(k-1)^2\psi(\varepsilon)^2|G'|^2. \tag{17}$$

Combining (15) and (17), and $|G'| \sim \binom{n-1}{k-1}$, we obtain

$$|P_1(G')| \lesssim \psi(\varepsilon)\cdot\frac{1}{2}(k-1)|G'|\binom{n-1}{k-2} \lesssim \psi(\varepsilon)|P(K)|.$$

This proves Claim 2.

We now complete the proof for $k > 3$. Since $|G'| \sim |K|$, it is straightforward to see that

$|P(G')| \sim |P(K)|$. By (9), $t \leq kn^{\alpha}$ where $\alpha < \frac{1}{2}$ (this relies on $k > 3$). Therefore

$$\begin{aligned} |P(K)| \ &\sim\ |P(G')| \ = \ |P_1(G')| + |P_2(G')| \\ &\lesssim\ \psi(\varepsilon)|P(K)| + 2\binom{t}{2}\binom{2k-4}{k-2}\binom{n}{2k-4} \\ &\sim\ \psi(\varepsilon)|P(K)|. \end{aligned} \tag{18}$$

However, $\psi(\varepsilon) < 1$, so the above inequality is a contradiction.

For $k = 3$, G' is a graph and $P(G')$ is the set of paths of length two in G'. The problem with the above arguments for $k = 3$ is that (9) only gives $t \leq 3n^{7/11}$, which is too large for (18) to hold and provide a contradiction. Therefore we go one step further, and count paths of length three in G' instead of paths of length two. Let $P_3(G')$ be the number of paths of length three in G' with edges from three different G_xs. By Claim 2,

$$|P_2(G')| = |P(G')| - |P_1(G')| \gtrsim (1 - \psi(\varepsilon))|P(K)| \gg n^3. \tag{19}$$

As in Claim 1, if $\{\{e, f\}, \{e', f'\}\}$ is a type 2 quadrilateral of G and $e, e' \in G_x$ and $f, f' \in G_y$, then we obtain a two-regular subgraph of H. So each type 2 quadrilateral contains edges from at least three different G_xs, and these edges form a path of length three in G'. Consequently, as in (15), the convexity of binomial coefficients and (19) give

$$|P_3(G')| \geq \frac{1}{4}|Q_2(G')| \geq \frac{1}{4}\binom{|P_2(G')|/N}{2}N \gg n^4$$

since $N = \binom{n-1}{2}$. Let (A, B) be a random partition of $V(G')$, defined by placing a vertex in A with probability $\frac{1}{2}$ and in B with probability $\frac{1}{2}$, independently for each vertex of $V(G')$. Let G^* denote the graph consisting of all edges between A and B. Then the expected value of $|P_3(G^*)|$ is exactly $\frac{1}{8}|P_3(G')|$, so there is a partition of G' for which

$$|P_3(G^*)| \geq \frac{1}{8}|P_3(G')| \gg n^4. \tag{21}$$

Let $e_1e_2e_3$ and $f_1f_2f_3$ be two paths in G^* with the same pair of endpoints. Suppose $e_i \in G_{j(i)}$ and $f_i \in G_{h(i)}$ where $\{j(1), j(2), j(3)\} = \{h(1), h(2), h(3)\}$. Since G^* is bipartite, amongst these edges there is a cycle C of length four or six containing exactly zero or two edges from each $G_{j(i)}$, $i = 1, 2, 3$. It is easily checked that the unique edges of H' which contain the edges of C form a two-regular subgraph of H, which is a contradiction. We conclude that at most $\binom{t}{3}$ paths of length three in G^* with edges in different G_is have the same pair of endpoints. It follows that

$$|P_3(G^*)| \leq \binom{t}{3}\binom{n}{2} \ll n^{4-\frac{1}{11}}$$

using (9). This contradicts (21), and completes the proof of Theorem 4. □

Corollary 1. *Fix $k \geq 3$. For every $\varepsilon > 0$, there exists $n_{k,\varepsilon}$ such that if $n > n_{k,\varepsilon}$ and $H \in \mathcal{P}_k(n)$ with $|H| \geq \binom{n-1}{k-1}$, then there is a vertex $v \in V(H)$ such that $|H - \{v\}| < \varepsilon n^{k-1}$.*

Proof. Suppose for contradiction that there exists $\varepsilon > 0$ and a sequence $H_{n_i} \in \mathcal{P}_k(n_i)$ for $i \geq 1$ with $|H_{n_i}| \geq \binom{n_i-1}{k-1}$ and $|H_{n_i} - \{v\}| \geq \varepsilon n^{k-1}$ for every $v \in V(H_{n_i})$. Now consider a sequence $G_l \in \mathcal{P}_k(l)$ satisfying $G_l = H_{n_i}$ when $l = n_i$ and $|G_l| \sim \binom{n-1}{k-1}$ (one can easily construct such a sequence by letting G_l be a star for $l \neq n_i$). Then Theorem 4 implies that $\Delta(G_l) \sim \binom{n-1}{k-1}$, and passing to the subsequence H_{n_i} we conclude that $\Delta(H_{n_i}) \sim \binom{n_i-1}{k-1}$ which is a contradiction. $\square$

6 The Exact Result

In this section we prove Theorem 1. Our main tools are the asymptotic and stability result. Let $H \in \mathcal{P}_k(n)$, where $k \geq 4$ is even, and suppose $|H| = \binom{n-1}{k-1}$. Let $\varepsilon = \frac{1}{100k^{4k}}$. By Corollary 1, for large enough n, there is a vertex $v \in V(H)$ such that

$$|H - \{v\}| \leq \varepsilon n^{k-1}. \tag{23}$$

Let $H^* = H - \{v\}$. To complete the proof, we show $|H^*| = 0$. Suppose, for a contradiction, that $|H^*| > 0$. For $|e| = k-2$, let $d_v(e)$ be the degree of a set e in H_v.

Claim 1. There are pairwise disjoint $(k-2)$-sets $e_1, e_2, \ldots, e_k \subset V(H)\backslash\{v\}$ such that for $i \in \{1, 2, \ldots, k\}$,

$$d_v(e_i) \geq n - k + 1 - \frac{2k|H^*|}{\binom{n-1}{k-2}}.$$

Proof. Let $d(n)$ be the lower bound in the claim, F the family of $(k-2)$-sets in $V(H_v)$ whose degree is at least $d(n)$, and let F^c be the rest of the $(k-2)$-sets in $V(H_v)$. Then

$$(k-1)|H_v| = \sum_e d_v(e) \leq |F|(n-k+1) + |F^c|d(n).$$

where the sum is over $e \subset V(H_v)$ of size $k-2$. As $|F| + |F^c| = \binom{n-1}{k-2}$, this implies

$$\frac{2k|H^*||F|}{\binom{n-1}{k-2}} \geq (k-1)|H_v| - d(n)\binom{n-1}{k-2} = 2k|H^*| - (k-1)|H^*|$$

since $|H^*| = \binom{n-1}{k-1} - |H_v|$. Hence $|F| \geq \left(1 - \frac{k-1}{2k}\right)\binom{n-1}{k-2} > \frac{1}{2}\binom{n-1}{k-2}$. Let $\{e_1, e_2, \ldots, e_l\}$ be a maximum matching in F. If $l < k$, then all other sets of F have an element within $e_1 \cup e_2 \cup \cdots \cup e_l$, which implies (since we may take n is large enough) that

$$|F| \leq (k-1)(k-2)\binom{n-1}{k-3} < k^2\binom{n-1}{k-3} < \frac{1}{2}\binom{n-1}{k-2}.$$

This contradiction shows that $l \geq k$ and the claim is proved.

Let $W = \{w \in V(H_v) \mid \exists i : e_i \cup \{v, w\} \notin H\}$. By Claim 1, $|W| < k(n - d(n))$. By adding points arbitrarily to W, we may assume that $|W| = \lceil k(n - d(n)) \rceil$. Define, for each $i \in \{0, 1, \ldots, k\}$, $H_i = \{e \in H^* : |e \cap W| = i\}$ and let $G = H_0 \cup H_1 \cup \cdots \cup H_{k-2}$. Note that the H_i partition H^*.

Claim 2. $|H_{k-1}| \le \binom{|W|}{k-1}$.

Proof. Suppose there exists a $(k-1)$-set $e \subset W$ and elements $y, z \notin W$ such that $e \cup \{y\}, e \cup \{z\} \in H_{k-1}$. Since $|e| = k-1$, by Claim 1 there exists i such that $e_i \cap e = \emptyset$ and $e_i \cup \{v, y\}, e_i \cup \{v, z\} \in H$. Together with $e \cup \{y\}$ and $e \cup \{z\}$, this yields a two-regular subgraph in H. This contradiction implies that we may count sets in H_{k-1} by their intersection with W to obtain $|H_{k-1}| \le \binom{|W|}{k-1}$.

Claim 3. $|H^*| \ge \binom{n-k-1}{k/2-1}$.

Proof. Since $|H^*| \ge 1$, there exists $e \in H^*$. Let e' be a $\frac{k}{2}$-subset of e. Now for each choice of a $(\frac{k}{2}-1)$-set $f \subset V(H_v) \backslash e$, one of the sets $f \cup e' \cup \{v\}$ or $f \cup (e \backslash e') \cup \{v\}$ must be missing from H, otherwise these two sets together with e form a two-regular subgraph of H. Consequently, $|H^*| \ge \binom{n-k-1}{k/2-1}$.

Claim 4. $|G| > \frac{99}{100}|H^*|$.

Proof. We show $|H_{k-1}| + |H_k| < \frac{1}{100}|H^*|$. By Theorem 3, there is a smallest integer $n_0 = n_0(k)$ such every k-graph in $\mathcal{P}_k(n)$ with $n > n_0$ has at most $2\binom{n_0-1}{k-1}$ edges. Assume also that $n_0 > 3k^2$. If $|W| < n_0$, then $|H_k| + |H_{k-1}| < |W|^k < n_0^k$. If n is large enough then, by Claim 3, this is less than $\frac{|H^*|}{100}$, as required. So we assume $|W| > n_0$. Since the k-graph H_k itself contains no two-regular subgraph, $|H_k| \le 2\binom{|W|-1}{k-1}$. Recalling that $|W| = \lceil k(n - d(n)) \rceil$, and using $|W| > n_0 > 3k^2$, we obtain

$$|W| < k^2 + \frac{2k^2|H^*|}{\binom{n-1}{k-2}} < \frac{3}{2}\frac{2k^2|H^*|}{\binom{n-1}{k-2}} = \frac{3k^2|H^*|}{\binom{n-1}{k-2}}.$$

Now suppose, for a contradiction, that $|H_k| + |H_{k-1}| > \frac{|H^*|}{100}$. By Claim 2,

$$\frac{|H^*|}{100} < |H_k| + |H_{k-1}| < 2\binom{|W|-1}{k-1} + \binom{|W|}{k-1} < \frac{3|W|^{k-1}}{(k-1)!} < \frac{k^{2k}|H^*|^{k-1}}{\binom{n-1}{k-2}^{k-1}}.$$

Simplifying,

$$|H^*|^{k-2} > \binom{n-1}{k-2}^{k-1} \frac{1}{100k^{2k}} > \left(\frac{n-1}{k-2}\right)^{(k-2)(k-1)} \frac{1}{100k^{2k}} > \frac{(n-1)^{(k-2)(k-1)}}{100k^{(k-2)(k-1)+2k}}.$$

This implies that $|H^*| > \frac{(n-1)^{k-1}}{100k^{k-1+2k/(k-2)}} > \varepsilon n^{k-1}$, which contradicts (23). This completes the proof of Claim 4.

Let p be the number of pairs (e, f) such that

(1) $v \notin e \in H$ and $|e \cap W| \le k-2$ (i.e. $e \in \cup_{i=0}^{k-2} H_i$)
(2) $v \in f \notin H$ and $|f| = k$ (so the number of such fs is $|H^*|$)
(3) $|e \cap f| = \frac{k}{2}$
(4) $e \cap f$ and $e \backslash f$ (which are $\frac{k}{2}$-sets) have a point outside W.

Fix $e \in H$ as above. Suppose that g is a $\frac{k}{2}$-subset of e such that neither g nor $e \backslash g$ lies within W. Let h be a $(\frac{k}{2}-1)$-subset of $V(H) \backslash (W \cup e \cup \{v\})$. Then the three sets $e, g \cup h \cup \{v\}, (e \backslash g) \cup h \cup \{v\}$ form a two-regular subgraph. Consequently, either $g \cup h$ or $(e \backslash g) \cup h$ is not in H_v. The number of pairs $\{g, e \backslash g\}$ that we can take is at least

$$\frac{1}{2}\binom{k}{k/2} - \binom{|W \cap e|}{k/2} \geq \frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2}.$$

Therefore, counting p from the e's we have

$$\begin{aligned} p &\geq (0.99)|H^*| \left(\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2} \right) \binom{n - |W| - k - 1}{k/2 - 1} \\ &> (0.98)|H^*| \left(\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2} \right) \binom{n}{k/2 - 1}, \end{aligned}$$

where the last inequality holds since $|W| < \varepsilon k^{4k} n$ and n is sufficiently large.

On the other hand, counting p from the fs we have $p \leq |H^*|\binom{k-1}{k/2} q$, where q is the number of times the $\frac{k}{2}$-sets $g \subset f \backslash \{v\}$ can extend to e, where $e \cap f = g$. Let F be the $\frac{k}{2}$-graph of these possible extensions of g to e. Let $F_0 \subset F$ be the $\frac{k}{2}$-graph whose edges have no points in W and $F_1 \subset F$ be the $\frac{k}{2}$-graph whose edges have at least one point in W.

Claim 5. $|F_0| < 2\binom{n}{k/2-1}/k$ and $|F_1| \leq \varepsilon k^{2k}\binom{n}{k/2-1}$.

> *Proof.* We start with F_1: to each $\frac{k}{2}$-set $h \in F_1$ associate a $(\frac{k}{2} - 1)$-set $h' \subset h$ such that $h' \cap W \neq \emptyset$ and $W \cap h \subset h'$. Such an h' exists by the definition of F_1. If there are distinct $h_1, h_2 \in F_1$ with $h_1' = h_2'$, then there are distinct vertices $y, z \notin W$ such that $h_1 = h_1' \cup \{y\}$ and $h_2 = h_2' \cup \{z\}$ By Claim 1, there exists i for which e_i has no point of $W \cap (g \cup h_1 \cup h_2)$. Now the four sets $g \cup h_1, g \cup h_2, e_i \cup \{v, y\}, e_i \cup \{v, z\}$ form a two-regular subgraph of H, contradicting $H \in \mathcal{P}_k(n)$. Consequently, $|F_1|$ is at most the number of $(\frac{k}{2} - 1)$-sets of $V(H)$ that contain at least one point of W. This is at most
>
> $$|W|\binom{n}{k/2-2} < \frac{3k^2 \varepsilon n^{k-1}}{\binom{n-1}{k-2}}\binom{n}{k/2-2} < \varepsilon k^{2k}\binom{n}{k/2-1}.$$
>
> This gives the bound on $|F_1|$. If there are distinct $h_1, h_2 \in F_0$ with $|h_1 \cap h_2| = k/2 - 1$, then arguing as above we find two-regular subgraph of H. Consequently, $|F_0| < \binom{n}{k/2-1}/\binom{k/2}{k/2-1}$.

Putting these bounds together we have $q \leq \varepsilon k^{2k}\binom{n}{k/2-1} + 2\binom{n}{k/2-1}/k$, and this gives

$$p \leq (1 + \varepsilon k^{4k})|H^*|\binom{k-1}{k/2}\frac{2}{k}\binom{n}{k/2-1}.$$

Comparing the upper and lower bounds for p and dividing by $|H^*|\binom{n}{k/2-1}$ yields

$$(0.98)\left(\frac{1}{2}\binom{k}{k/2} - \binom{k-2}{k/2}\right) < (1 + \varepsilon k^{4k})\frac{2}{k}\binom{k-1}{k/2}.$$

Since $\varepsilon < k^{4k}/100$ this implies that

$$(0.97)\left(\frac{1}{2}\binom{k}{k/2}-\binom{k-2}{k/2}\right)<\frac{2}{k}\binom{k-1}{k/2}.$$

A short calculation shows that this is equivalent to $(0.97k-2)(k-1) < 0.97k(\frac{k}{2}-1)$, and it is easily verified that this is false for $k \geq 4$. This contradiction completes the proof of the theorem. $\square$

7 Concluding Remarks

A k-graph is r-regular if all its vertices have degree r. In contrast to Theorem 1, if the degrees in a k-graph are all the same, then a linear number of edges already forces a two-regular subgraph. Precisely, let ϕ_k denote the maximum number ϕ such that there exists a ϕ-regular k-graph containing no two-regular subgraph. Then Lemma 1 immediately implies that $\phi_k \leq (ek)^k$. On the other hand, we have a lower bound of $\binom{3k/2-1}{k-1}$ when k is even and $\binom{2k-1}{k-1}$ when k is odd, by taking complete k-graphs of the appropriate size (these contain no two-regular subgraphs because every two-regular subgraph has at least $3k/2$ vertices when k is even and at least $2k$ vertices when k is odd). The lower bounds are of order c^k, so there is a substantial gap in the bounds for ϕ_k. We leave the open problem of determining ϕ_k and, in particular, ϕ_3. It is expected that if a k-graph is ϕ-regular and ϕ is a large enough constant depending on k and r, then every ϕ-regular k-graph has an r-regular subgraph (a subgraph in which every vertex has degree r). In fact, this should hold for multi-k-graphs – instead of a set of edges a multiset of edges is allowed. Therefore we make the following conjecture:

Conjecture 2. *Let $k, r \geq 2$. There exists an integer $\phi_k(r)$ such that for $\phi > \phi_k(r)$, every ϕ-regular multi-k-graph contains an r-regular subgraph.*

This conjecture is wide open for $k, r \geq 3$. If we superimpose $r-1$ copies of the complete k-graph on $k+1$ vertices, and k and r are relatively prime, then we obtain a multi-k-graph containing no r-regular subgraph. This simple construction shows that if $\phi_k(r)$ exists, then $\phi_k(r) \geq k(r-1)$. For $k = 2$, in other words, for multigraphs, Tâskinov [15] completely determined $\phi_k(r)$ using Tutte's f-Factor Theorem. Unfortunately, there is no analogous theorem for k-graphs when $k \geq 3$. The following positive evidence for Conjecture 2 follows immediately by extending the proof of Alon, Friedland, Kalai [1] and uses Chevalley's theorem:

Theorem 5. *Let H be an n-vertex k-graph, such that H is $k(r-1)+1$-regular, where r is a prime number. Then H has a subgraph all of whose vertex degrees are elements of $\{r, 2r, \ldots, (k-1)r\}$.*

The construction of superimposed multi-k-graphs given above shows that Theorem 5 is tight whenever r is a prime which does not divide k. Further evidence for Conjecture 2 comes from Rödl's packing method [13]. A k-graph is linear if no two of its edges intersect in two or more points. If M is a matching in a k-graph H, let $\mathrm{ex}(M)$ denote the number of vertices not covered by M. Rödl's Theorem [13] says that every linear n-vertex d-regular k-graph contains a matching M such that $\mathrm{ex}(M) \leq d^{-\varepsilon}n$, for some constant $\varepsilon > 0$ depending only on k. In fact, the degrees of the vertices in the hypergraph are allowed to be between $(1-\delta)d$ and $(1+\delta)d$

for the same conclusion, provided $\delta > 0$ is a sufficiently small constant depending on ε. By repeatedly removing r such matchings from a linear d-regular k-graph, we see that for any fixed r, we obtain a subgraph in which all vertices have degree at most r, and at most $rd^{-\varepsilon}n$ vertices have degree less than r. In other words, we find an "almost r-regular" subgraph. On the other hand, we do not even have a verification that every large enough Steiner triple system has a three-regular subgraph.

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