Note – Implicit Differentiation

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A surface in $\mathbb{R}^3$ is represented explicitly as $z = f(x, y)$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$. It is not always the case, however, that we can write the surface as $z = f(x, y)$. For example, the unit sphere is represented most naturally by the equation $x^2+y^2+z^2 = 1$ and we could then attempt to solve for z as a function of x and y. However, it is clear that there is no single function $f(x, y)$ such that $z = f(x, y)$ defines a sphere since for each x and y value there are two z values on the sphere. Despite this, we can say that around any point on the sphere, we get a surface $z = f(x, y)$, and furthermore the surface is smooth.

A surface in $\mathbb{R}^3$ is therefore often represented implicitly – that is there is an equation $g(x, y, z) = 0$ which defines the surface. For the sphere it was $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. While we might not be able to solve explicitly for z in terms of x and y, it is still reasonable to ask if parts of the surface are smooth and can be solved for z as a function of x and y – the key for this is the implicit function theorem, and then we can ask for equations of tangent planes and gradients to that surface – the key to that is implicit differentiation.

If we know that $g(x_1, x_2, \dots, x_n, z) = 0$ defines a smooth hypersurface $z = f(x_1, x_2, \dots, x_n)$ at a point $a \in \mathbb{R}^{n+1}$, for example using the implicit function theorem, then we can ask for the equation of the tangent plane at a. However, we can't proceed directly since in general we don't know what f is. The key is implicit differentiation: we assume $z = f(x_1, x_2, \dots, x_n)$ and differentiate $g(x_1, x_2, \dots, x_n, z) = 0$ with respect to each x_i. The aim is to find $\frac{\partial f}{\partial x_i}$ for all i, for then we know the tangent plane at a is

$$z = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i)$$

from the formula in the lecture notes. For instance, suppose we want the tangent line to the curve $x^z + z^x = e + 1$ in $\mathbb{R}^2$ at the point $(1, 1)$. We implicitly assume $z = f(x)$ – note that near $(e, 1)$ the implicit function theorem says we can do this – and differentiate the equation:

$$\frac{d}{dx}(x^z + x^z) = \frac{d}{dx}(2) \Rightarrow x^z\Big(\frac{z}{x} + \frac{df}{dx}\ln x\Big) + z^x\Big(\ln z + \frac{x}{z}\frac{df}{dx}\Big) = 0.$$

At the point $(e, 1)$ we get

$$e^1\Big(\frac{1}{e} + \frac{df}{dx}\Big) + \Big(\frac{e}{1}\frac{df}{dx}\Big) = 0$$

and therefore

$$\frac{df}{dx} = -\frac{1}{2e}.$$

That means the equation of the tangent line is

$$z = 1 - \frac{1}{2e}(x - e) = \frac{3}{2} - \frac{1}{2e}x.$$

In fact we can find a formula for $\frac{df}{dx}$ given an equation $g(x, z) = 0$: the chain rule gives

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{df}{dx} = 0$$

and so

$$\frac{df}{dx} = -\frac{g_x}{g_z}.$$

This can be generalized to more than two variables.