A. Summary of Methods of Integration

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• Fundamental Theorem I: if F is an antiderivative of f, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

• Fundamental Theorem II: if F is an antiderivative of f, then

$$\frac{d}{dt}\int_a^t f(x)\ dx = F(t).$$

• Basic integrals

$\int e^x\ dx = e^x + c$	$\int x^r\ dx = \frac{1}{r+1}x^{r+1} + c\ (r \neq -1)$	$\int 1/x\ dx = \ln\|x\| + c$
$\int \sin x\ dx = -\cos x + c$	$\int \cos x\ dx = \sin x + c$	$\int \tan x\ dx = \ln\|\sec x\| + c$
$\int 1/(x^2+1)\ dx = \arctan x + c$	$\int 1/\sqrt{1-x^2}\ dx = \arcsin x + c$	$\int \ln x\ dx = x\ln x - x + c$

• Substitution rule: if $x = g(u)$ then

$$\int f(x)dx = \int f(g(u))g'(u)du.$$

• Integration by parts: if $G(x)$ is an antiderivate of $g(x)$, then

$$\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx$$

• Long division: if $P(x)$ and $Q(x)$ are polynomials and $\deg P(x) \geq \deg Q(x)$, then we may perform long division of $Q(x)$ into $P(x)$ to evaluate $\int P(x)/Q(x)\ dx$. Partial Fractions: if $Q(x) = (x-a)(x-b)$ where $a \neq b$, then for some constants A and B,

$$\frac{1}{Q(x)} = \frac{A}{x-a} + \frac{B}{x-b}$$

• Improper integrals: these are defined as limits

$$\int_{\infty}^{\infty} f(x)\ dx = \lim_{b\to\infty} f(b) - \lim_{a\to-\infty} f(a).$$

If f has a vertical asymptote at $x = c$, then

$$\int_a^b f(x)\ dx = \lim_{s\to c^-}\int_a^s f(x)\ dx + \lim_{t\to c^+}\int_t^b f(x)\ dx.$$

• Comparison test:

(1) Let $f(x)$ and $g(x)$ be continuous functions such that $0 \leq f(x) \leq g(x)$ for $a \leq x \leq b$. If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.

(2) Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x) \geq 0$ for $a \leq x \leq b$. If $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

B. Summary of Applications

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• Volume formula: if a curve $y = f(x)$ is rotated around the x-axis for $a \leq x \leq b$, then the volume enclosed is

$$\int_a^b \pi f(x)^2 \, dx$$

• Arc length formula: the length of the curve $y = f(x)$ for $a \leq x \leq b$ is given by

$$\int_a^b \sqrt{1 + f'(x)^2} \, dx$$

• Polar co-ordinates: the area swept out by a curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ in polar co-ordinates is

$$\frac{1}{2}\int_\alpha^\beta f(\theta)^2 \, d\theta$$

The length of the curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ in polar co-ordinates is given by

$$\int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2} \, d\theta$$

• The center of mass of an area of uniform density enclosed by the curve $y = f(x)$ and $y = -f(x)$ for $a \leq x \leq b$ is given by

$$\frac{\int_a^b x f(x) \, dx}{\int_a^b f(x) \, dx}.$$

• A function $p(x) : \mathbb{R} \to \mathbb{R}$ is a probability density function if

$$\begin{array}{ll} (1) & 0 \leq p(x) \leq 1 \\ (2) & \int_{-\infty}^{\infty} p(x) \, dx = 1 \end{array}$$

The expectation or average value of x, if it exists, is given by

$$\int_{-\infty}^{\infty} x p(x) \, dx$$

• Work and force: The force exerted on an object of mass m with acceleration a is $m \cdot a$ parallel to the direction of motion. The work done is the product of the force exerted and the distance the object is moved.

• Computing work on solid objects: a solid is obtained by rotating $y = f(x)$ around the x-axis for $a \leq x \leq b$. If the mass density of the object is ρ, and the object is lifted through height of h, then the work done is

$$\int_0^h \pi \rho f(x)^2 \, dx.$$

• Separable equations: differential equation $\frac{dy}{dx} = f(x, y)$ is separable if $f(x, y) = g(x)h(y)$ for some functions g and h. In this case it is solved by dividing by $h(y)$ and multiplying by dx and then integrating both sides.

• Half-life: the equation $\frac{dy}{dx} = \alpha y$ models radioactive decay. The half-life, which is the time taken for the material to halve, is $-(\log 2)/\alpha$. The solution to this equation is $y = Ce^{\alpha x}$ for some constant C. The value of C is the value of y at time zero.