A. Summary of Methods of Integration

• Fundamental Theorem I: if $F$ is an antiderivative of $f$, then
  \[ \int_a^b f(x) \, dx = F(b) - F(a) \]

• Fundamental Theorem II: if $F$ is an antiderivative of $f$, then
  \[ \frac{d}{dt} \int_a^t f(x) \, dx = F(t). \]

• Basic integrals
  \[ \int e^x \, dx = e^x + c \]
  \[ \int x^r \, dx = \frac{1}{r+1}x^{r+1} + c \quad (r \neq -1) \]
  \[ \int 1/x \, dx = \ln |x| + c \]
  \[ \int \sin x \, dx = -\cos x + c \]
  \[ \int \cos x \, dx = \sin x + c \]
  \[ \int \tan x \, dx = \ln |\sec x| + c \]
  \[ \int 1/\sqrt{1-x^2} \, dx = \arcsin x + c \]
  \[ \int \ln x \, dx = x \ln x - x + c \]

• Substitution rule: if $x = g(u)$ then
  \[ \int f(x) \, dx = \int f(g(u))g'(u) \, du. \]

• Integration by parts: if $G(x)$ is an antiderivative of $g(x)$, then
  \[ \int f(x)g(x) \, dx = f(x)G(x) - \int f'(x)G(x) \, dx \]

• Long division: if $P(x)$ and $Q(x)$ are polynomials and $\deg P(x) \geq \deg Q(x)$, then we may perform long division of $Q(x)$ into $P(x)$ to evaluate $\int P(x)/Q(x) \, dx$. Partial Fractions: if $Q(x) = (x-a)(x-b)$ where $a \neq b$, then for some constants $A$ and $B$,
  \[ \frac{1}{Q(x)} = \frac{A}{x-a} + \frac{B}{x-b} \]

• Improper integrals: these are defined as limits
  \[ \int_{-\infty}^{\infty} f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx - \lim_{a \to -\infty} f(a). \]

If $f$ has a vertical asymptote at $x = c$, then
\[ \int_a^b f(x) \, dx = \lim_{s \to c^-} \int_a^s f(x) \, dx + \lim_{t \to c^+} \int_t^b f(x) \, dx. \]

• Comparison test:
  (1) Let $f(x)$ and $g(x)$ be continuous functions such that $0 \leq f(x) \leq g(x)$ for $a \leq x \leq b$. If $\int_a^b g(x) \, dx$ converges, then $\int_a^b f(x) \, dx$ converges.
  (2) Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x) \geq 0$ for $a \leq x \leq b$. If $\int_a^b g(x) \, dx$ diverges, then $\int_a^b f(x) \, dx$ diverges.
B. Summary of Applications

- Volume formula: if a curve $y = f(x)$ is rotated around the $x$-axis for $a \leq x \leq b$, then the volume enclosed is
  $$\int_a^b \pi f(x)^2 \, dx$$

- Arc length formula: the length of the curve $y = f(x)$ for $a \leq x \leq b$ is given by
  $$\int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

- Polar co-ordinates: the area swept out by a curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ in polar co-ordinates is
  $$\frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 \, d\theta$$

  The length of the curve $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$ in polar co-ordinates is given by
  $$\int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + (f'(\theta))^2} \, d\theta$$

- The center of mass of an area of uniform density enclosed by the curve $y = f(x)$ and $y = -f(x)$ for $a \leq x \leq b$ is given by
  $$\frac{\int_a^b xf(x) \, dx}{\int_a^b f(x) \, dx}.$$

- A function $p(x) : \mathbb{R} \to \mathbb{R}$ is a probability density function if
  \begin{align*}
  (1) & \quad 0 \leq p(x) \leq 1 \\
  (2) & \quad \int_{-\infty}^{\infty} p(x) \, dx = 1
  \end{align*}

  The expectation or average value of $x$, if it exists, is given by
  $$\int_{-\infty}^{\infty} xp(x) \, dx$$

- Work and force: The force exerted on an object of mass $m$ with acceleration $a$ is $m \cdot a$ parallel to the direction of motion. The work done is the product of the force exerted and the distance the object is moved.

- Computing work on solid objects: a solid is obtained by rotating $y = f(x)$ around the $x$-axis for $a \leq x \leq b$. If the mass density of the object is $\rho$, and the object is lifted through height of $h$, then the work done is
  $$\int_{0}^{h} \pi \rho f(x)^2 \, dx.$$ 

- Separable equations: differential equation $\frac{dy}{dx} = f(x, y)$ is separable if $f(x, y) = g(x)h(y)$ for some functions $g$ and $h$. In this case it is solved by dividing by $h(y)$ and multiplying by $dx$ and then integrating both sides.

- Half-life: the equation $\frac{dy}{dx} = \alpha y$ models radioactive decay. The half-life, which is the time taken for the material to halve, is $-(\log 2)/\alpha$. The solution to this equation is $y = C e^{\alpha x}$ for some constant $C$. The value of $C$ is the value of $y$ at time zero.