Lecture 1 : Monday March 31st

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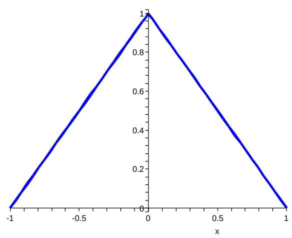
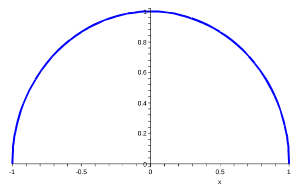
1.1 Areas under curves

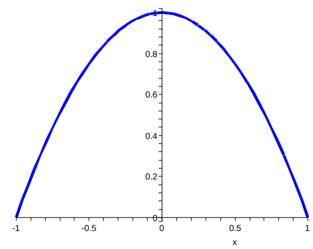
Consider the problem of finding the area between the given curve and the x-axis in each of the diagrams below.



$y = 1 - |x|$



$y = \sqrt{1 - x^2}$



$y = 1 - x^2$

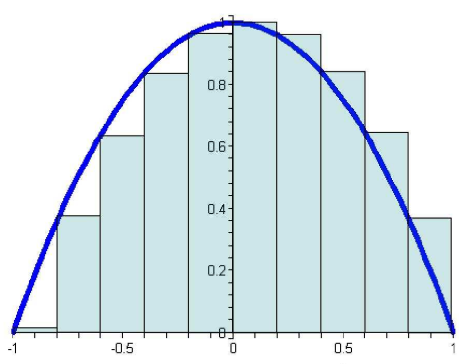
It is not hard to find the area between the x-axis and the curve in the first two graphs, since the curve $y = 1 - |x|$ defines a triangle and the curve $y = \sqrt{1 - x^2}$ defines a semicircle, and we have formulas for the areas of these shapes:

$$\text{Area}(\triangle) = 1 \qquad \text{Area}(\frown) = \frac{1}{2}\pi.$$

Now in the third picture, we have the parabola $f(x) = 1 - x^2$. We don't have a formula for the area of such a shape, so let's spend some time seeing how we could get an approximate area for this shape.

1.2 Left Riemann Sums

How could we get a good estimate of the area? It turns out that the actual area is $\frac{4}{3}$, and we will see how to work this out in a flash later on using integration. Since we know how to find the area of a rectangle, we could approximate the region by summing up the areas of rectangles as shown below. These sums are called Riemann sums.



Approximate area

The rectangles in the picture under the curve $f(x) = 1 - x^2$ are defined precisely as follows. First cut interval under the curve on the x-axis into ten intervals of equal length. Create rectangles $\Box_1, \Box_2, \ldots, \Box_{10}$ whose bases are these intervals and where the heights of the boxes are

$$\text{Ht}(\Box_1) = f(-1) = 0 \quad \text{Ht}(\Box_2) = f(-0.8) = 0.36 \quad \cdots \quad \text{Ht}(\Box_{10}) = f(0.8) = 0.36.$$

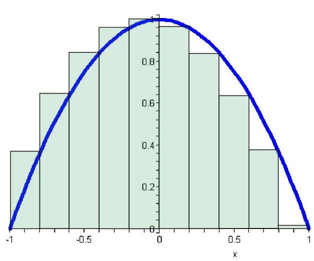
Notice that the height of $\Box_i$ is just the value of the function f at the left endpoint of the ith interval. To find the approximate area, we just add up the areas of the rectangles $\Box_i$ to get

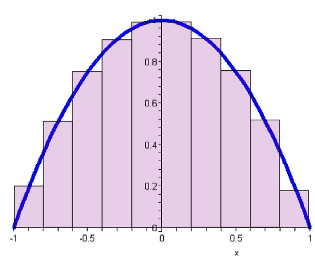
$$\begin{aligned} \text{Apx Area} &= \text{Area}(\Box_1) + \text{Area}(\Box_2) + \text{Area}(\Box_3) + \cdots + \text{Area}(\Box_{10}) \\ &= 0.2 \cdot (0 + 0.36 + 0.64 + 0.84 + 0.96 + 1 + 0.96 + 0.84 + 0.64 + 0.36) = 1.32 \end{aligned}$$

So the approximate area found using these rectangles is 1.32. For the reason that we took left endpoints to determine the heights of the rectangles, we call the above sum a left Riemann sum.

1.3 Right Riemann Sums

There isn't any reason to use the rectangles we gave above, where the height of $\Box_i$ was the value of f at the left endpoint of the ith interval. We could have chosen the height of $\Box_i$ to be f at the right endpoint. If we do that, then the height of $\Box_1$ is now $f(-0.8) = 0.36$, the height of $\Box_2$ is $f(-0.6) = 0.64$, and so on. The area is called a right Riemann sum. Alternatively, we could pick the midpoint of the ith interval and let the height of $\Box_i$ be the value of f at this midpoint to get another Riemann sum. The pictures of these rectangles are shown below:





Right Midpoint

Let's work out the approximate areas by finding the values of the Riemann sums: in the first picture

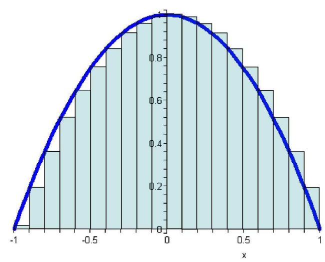
$$\text{Apx Area} = 0.2 \cdot (0.36 + 0.64 + 0.84 + 0.96 + 1 + 0.96 + 0.84 + 0.64 + 0.36 + 0) = 1.32$$

as expected since this is just a reflection of the pictures with the blue rectangles. In the second picture, where the height of the ith rectangle is $f(0.2i - 1.1)$, we get

$$\text{Apx Area} = 0.2 \cdot (0.36 + 0.64 + 0.84 + 0.96 + 1 + 0.96 + 0.84 + 0.64 + 0.36 + 0) = 1.34.$$

Now we have overestimated the area.

There is no reason to stick to ten rectangles. In fact, a key observation is that the more rectangles we use, the better approximation we will have for the areas. For instance, consider increasing the number of blue rectangles to twenty. The picture looks as follows:



Finer rectangular partitions

Now the intervals are $[-0.1i, -0.1i + 0.1]$ for $i = -10 \ldots 9$ and the height of the ith rectangle is $f(-0.1i)$. Let's see that we get a better gurss of the true area, which is $\frac{4}{3}$:

$$\text{Apx Area} = 0.1 \cdot (0 + 0.19 + 0.36 + \cdots + 1 + 0.91 + \cdots + 0.19) = 1.33.$$

This is already accurate to within two decimal places of the true area. Using a computer, we can get more and more accurate estimates using more and more rectangles. For example, the table below shows the better and better accuracy using left endpoint rectangles:

Number of rectangles	10	20	100	1000
Apx Area	1.32	1.33	1.3332	1.333332