Lecture 10 : Monday April 21st

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10.1 Trigonometric substitutions: Two basic examples

The purpose of trigonometric substitutions is to evaluate integrals of functions of $\sqrt{1-x^2}$ and $1+x^2$. This is based on the simple trig identities

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad \sec^2\theta = 1 + \tan^2\theta.$$

So if $x = \sin\theta$ then $\sqrt{1-x^2} = \cos\theta$ and if $x = \tan\theta$ then $1 + x^2 = \sec^2\theta$.

Example 1.
Evaluate the following integral

$$\int \frac{1}{\sqrt{1-x^2}}\, dx.$$

We let $x = \sin\theta$, as suggested above. By the substitution rule, we need to work out $\frac{dx}{d\theta} = \cos\theta$. Therefore, since $\sqrt{1-x^2} = \cos\theta$, the substitution rule gives

$$\int \frac{1}{\sqrt{1-x^2}}\, dx \;=\; \int \frac{1}{\cos\theta}\frac{dx}{d\theta} d\theta = \int \frac{\cos\theta}{\cos\theta} d\theta = \int 1 d\theta = \theta + c.$$

Since $x = \sin\theta$, $\theta = \arcsin x$ and therefore the integral is

$$\int \frac{1}{\sqrt{1-x^2}} dx \;=\; \arcsin x + c.$$

Example 2.
Evaluate the following integral

$$\int \frac{1}{1+x^2}\, dx.$$

We let $x = \tan\theta$ so that

$$\frac{dx}{d\theta} = \sec^2\theta.$$

By the substitution rule, and using $1 + x^2 = \sec^2\theta$, we get

$$\int \frac{1}{1+x^2}\, dx = \int \frac{1}{\sec^2\theta} \sec^2\theta d\theta = \int 1\ d\theta = \theta + c.$$

Since $x = \tan\theta$, we have $\theta = \arctan x$ and so

$$\int \frac{1}{1+x^2}\, dx \;=\; \arctan x + c.$$

10.2 Trigonometric substitutions: Further examples

Example 1.
The area of a semicircle with radius 1 is exactly

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

since the curve $y = \sqrt{1-x^2}$ for $-1 \leq x \leq 1$ is a semicircle. We expect the answer to be $\frac{1}{2}\pi$, since the area of the full circle is π. Now we prove this using trigonometric substitutions. In the integral, let $x = \sin\theta$ so that $\frac{dx}{d\theta} = \cos\theta$. By the substitution rule,

$$\int \sqrt{1-x^2}\, dx = \int \cos^2\theta d\theta.$$

This integral is still not so simple, but we can integrate by parts with both functions in the formula for integration by parts equal to $\cos\theta$. So the integral becomes

$$\int \cos^2\theta d\theta = -\sin\theta\cos\theta + \int \sin^2\theta d\theta.$$

Now $\sin^2\theta = 1 - \cos^2\theta$ so we get

$$\int \cos^2\theta d\theta = -\sin\theta\cos\theta + \int 1 d\theta - \int \cos^2\theta d\theta.$$

At first it looks like we again have to work out the integral we started with, since it appears again at the end of the equation. But if we let I be that integral, then we get

$$I = -\sin\theta\cos\theta + \int 1 d\theta - I.$$

This means

$$2I = -\sin\theta\cos\theta + \theta + c$$

in which case $I = \frac{1}{2}(\theta - \sin\theta\cos\theta) + c$. Now we remember that $\theta = \arcsin x$. If we put this into I, then we get an antiderivative $F(x)$ of $\sqrt{1-x^2}$, namely (ignoring c)

$$F(x) = \frac{1}{2}(\arcsin x - \sin\arcsin x \cos\arcsin x) = \frac{1}{2}(\arcsin x - x\cos\arcsin x).$$

What we really want is

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx.$$

By the first fundamental theorem of calculus, this is $F(1) - F(-1)$ where F is any antiderivative of $\sqrt{1-x^2}$. We already found an antiderivative, F, given above, so now we just have to find $F(1) - F(-1)$. Now $\arcsin 1 = \pi/2$ since $\sin(\pi/2) = 1$ and $\arcsin(-1) = -\pi/2$. Since $\cos(\pi/2) = \cos(-\pi/2) = 0$, we get

$$F(1) - F(-1) = \frac{1}{2}(\arcsin 1 - 1 \cos \arcsin 1) - \frac{1}{2}(\arcsin(-1) - (-1) \cos \arcsin(-1)) = \frac{\pi}{2}.$$

This agrees with the known formula for the area of a semicircle.

Example 2.
Suppose we want to find

$$\int_0^1 \frac{1}{\sqrt{4 - x^2}}\, dx.$$

If we put $x = \sin\theta$, then we get $\sqrt{4 - \sin^2\theta}$ in the denominator, which does not simplify as in Example 1 of the last section. Rather, we can put $x = 2\sin\theta$ to simplify the integral, and work it out as in that example. We have $\frac{dx}{d\theta} = 2\cos\theta d\theta$ and therefore the indefinite integral is

$$\int 1\, d\theta = \theta + c = \arcsin(x/2) + c.$$

Therefore $F(x) = \arcsin(x/2)$ is an antiderivative of $1/\sqrt{4 - x^2}$ which means that

$$\int_0^1 \frac{1}{\sqrt{4 - x^2}}\, dx = F(1) - F(0) = \arcsin(\frac{1}{2}) - \arcsin(0) = \frac{\pi}{6}.$$

This completes the example.