

12.1 Partial Fractions.

Example 1.

Evaluate

$$\int \frac{1}{(x^2 - 1)} dx.$$

This integral fits into partial fractions, since it is an integral of a rational function $P(x)/Q(x)$ where the degree of $P(x)$ is less than the degree of $Q(x)$ (so long division won't work). We observe that $Q(x) = (x - 1)(x + 1)$, and so

$$\frac{1}{Q(x)} = \frac{1}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{x}{x + 1}$$

where A and B are constants which we have to find. To find A and B , multiply both sides by $(x - 1)(x + 1)$ to get

$$1 = A(x + 1) + B(x - 1).$$

We group together like terms in this equation to get

$$1 = (A + B)x + A - B.$$

Since there is no x on the left hand side, the coefficient of x on the right hand side must also be zero. So $A + B = 0$. Also, since there is a 1 on the left, the constant term, $A - B$, must be equal to this, so $A - B = 1$. These simultaneous equations give $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. So our conclusion is that

$$\frac{1}{x^2 - 1} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1}.$$

So

$$\int \frac{1}{x^2 - 1} dx = \frac{1}{2} \int \frac{1}{x - 1} dx - \frac{1}{2} \int \frac{1}{x + 1} dx.$$

Therefore

$$\int \frac{1}{x^2 - 1} dx = \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + c.$$

We can simplify this a bit more if we want, since

$$\frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right|.$$

Example 2.

Evaluate

$$\int \frac{1}{x^2 + x + 1} dx.$$

We observe that $Q(x) = x^2 + x + 1$ does not factorize. So we complete the square:

$$x^2 + x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}.$$

The integral is now

$$\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx.$$

At this point we can make the substitution $w = x + \frac{1}{2}$. Then the integral becomes

$$\int \frac{1}{w^2 + \frac{3}{4}} dw.$$

Then we can make the trig substitution $w = \sqrt{\frac{3}{4}} \tan \theta$. The reason is that $w^2 + \frac{3}{4} = \frac{3}{4} \sec^2 \theta$ by the trig identities, and then the integral will simplify. We know that $dw = \sqrt{\frac{3}{4}} \sec^2 \theta$ and so the integral becomes

$$\int \sqrt{\frac{4}{3}} d\theta = \sqrt{\frac{4}{3}} \theta + c.$$

Since

$$\theta = \arctan \sqrt{\frac{4}{3}} w = \arctan \left(\sqrt{\frac{4}{3}} \left(x + \frac{1}{2}\right) \right)$$

we are done.

For such examples, make sure that you know how to complete the square: given a quadratic polynomial $x^2 + ax + b$, the formula is $(x + a/2)^2 + (b - a^2/4)$, and you can check by expanding the brackets.

Example 3.

Evaluate

$$\int \frac{\sqrt{x}}{x + \sqrt{x} + 1} dx.$$

This looks really unfamiliar, until we make the substitution $w = \sqrt{x}$. Then $x = w^2$ so $dx = 2w dw$. It follows that

$$\int \frac{\sqrt{x}}{x + \sqrt{x} + 1} dx = \int \frac{2w^2}{w^2 + w + 1} dw.$$

Since the integrand has the form $P(w)/Q(w)$, and the degree of $P(w)$ is at least the degree of $Q(w)$, we can do long division. After long division, we see

$$\frac{2w^2}{w^2 + w + 1} = 2 - \frac{2(w + 1)}{w^2 + w + 1}.$$

The derivative of $w^2 + w + 1$ is $2w + 1$. This is almost what is in the numerator, except it is $2w + 2$. So it makes sense to write

$$2 - \frac{2(w+1)}{w^2+w+1} = 2 - \frac{2w+1}{w^2+w+1} - \frac{1}{w^2+w+1}.$$

So we get

$$\int \frac{\sqrt{x}}{x + \sqrt{x} + 1} dx = \int \left(2 - \frac{2w+1}{w^2+w+1} - \frac{1}{w^2+w+1} \right) dw.$$

This integral can be evaluated. In fact, in the last example, we saw

$$\int \frac{1}{w^2+w+1} dw = \sqrt{\frac{4}{3}} \arctan\left(w - \frac{1}{2}\right) + c.$$

Using this information, we get

$$\int \left(2 - \frac{2w+1}{w^2+w+1} - \frac{1}{w^2+w+1} \right) dw = 2w - \ln|w^2+w+1| - \sqrt{\frac{4}{3}} \arctan\left(\sqrt{\frac{4}{3}}\left(w + \frac{1}{2}\right)\right) + c.$$

Note here that

$$\int \frac{2w+1}{w^2+w+1} dw = \int \frac{1}{v} dv = \ln|v| + c = \ln|w^2+w+1| + c$$

where $v = w^2 + w + 1$, and this is why the integral is $\ln|w^2 + w + 1| + c$.

12.3 A summary of methods

The main tools we have encountered so far in integration are as follows. It is important to be skillful with applying these rules together to be good at integration.

substitution rule

integration by parts

trigonometric substitutions

long division

partial fractions

completing the square

It is important to note that there is often more than one way to do a given integral, and that certain integrals just cannot be evaluated explicitly. For example, known integrals which can't be evaluated explicitly are

$$\int e^{x^2} dx \quad \int \frac{e^x}{x} dx \quad \int \sin(x^2) dx \quad \int \frac{\sin x}{x} dx \quad \int \sqrt{1+x^4} dx.$$