Lecture 13 : Wednesday April 30th

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13.1 Improper Integrals.

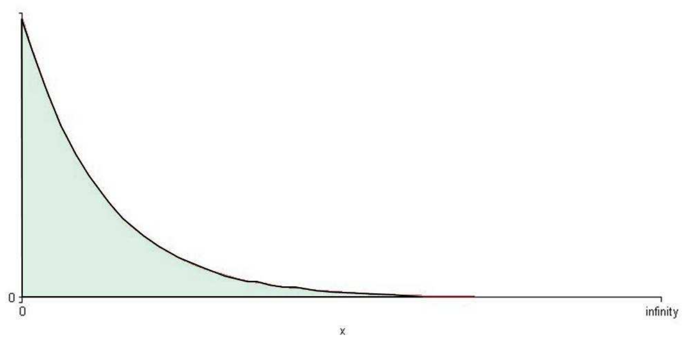
Suppose we determine the area under the curve $y = e^{-x}$ for $0 \leq x \leq t$. By definition, the area is

$$\int_0^t e^{-x}\, dx = 1 - e^{-t}$$

using the fundamental theorem of calculus. If we let t tend to infinity, we get

$$\lim_{t \to \infty} \int_0^t e^{-x}\, dx = 1.$$

This may be defined to be the area under the curve $y = e^{-x}$ and above the positive x-axis, and is called an improper integral, as illustrated below:



Improper integral as area

Instead of defining improper integrals explicitly as limits, we rewrite it more simply as

$$\int_a^\infty f(x)\, dx := \lim_{b \to \infty} \int_a^b f(x)\, dx.$$

so the upper limit of integration is ∞. These improper integrals appear in many contexts, especially in statistics and probability, and in physics.

To work out an improper integral $\int_a^\infty f(x)\, dx$, we use the first fundamental theorem to write

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where F is an antiderivative of f, and then take the limit as b tends to infinity:

$$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} F(b) - F(a).$$

There is no reason only to consider one limit of the improper integral being infinite. For example,

$$\int_{-\infty}^{b} f(x)\, dx = F(b) - \lim_{a\to-\infty} F(a)$$

and we can even have both limits being infinite

$$\int_{-\infty}^{\infty} f(x)\, dx = \lim_{b\to\infty} F(b) - \lim_{a\to-\infty} F(a).$$

Before going any further, let's do an example.

13.2 Example of improper integrals.

A probability density function in statistics is a function whose definite integral over the range of the function equals one. Let us check that $1/\pi(x^2+1)$ and $\frac{1}{2}e^{-|x|}$ are both probability density functions for $-\infty \le x \le \infty$. First,

$$\int_{-\infty}^{\infty} \frac{1}{\pi(x^2+1)}\, dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx = \frac{1}{\pi}\left(\lim_{b\to\infty} \arctan b - \lim_{a\to-\infty} \arctan a\right).$$

Now we know that $\arctan x$ has asymptotes at $x = \pm\pi/2$, so the first limit is $\pi/2$ and the second is $-\pi/2$. It follows that

$$\int_{-\infty}^{\infty} \frac{1}{\pi(x^2+1)}\, dx = 1$$

and so $1/\pi(x^2+1)$ is a probability density. Next we check that $\frac{1}{2}e^{-|x|}$ is too. We know that

$$\int_{-\infty}^{\infty} e^{-|x|}\, dx = \int_{0}^{\infty} e^{-x}\, dx + \int_{-\infty}^{0} e^{x}\, dx$$

by the basic rules of integration. Now

$$\int_{0}^{\infty} e^{-x}\, dx = 1$$

as we worked out before. Similarly,

$$\int_{-\infty}^{0} e^{x}\, dx = 1$$

and therefore

$$\int_{-\infty}^{\infty} e^{-|x|}\, dx = 2.$$

So $\frac{1}{2}e^{-|x|}$ is a probability density function.

Improper integrals need not always be finite, since the limits involved may not exist. For example,

$$\int_1^\infty \frac{1}{x} dx = \lim_{b\to\infty} \ln|x| - \ln 1 = \infty.$$

In this situation, we say that the integral diverges. Otherwise the integral converges.

12.3 Other improper integrals

In all the integrals so far, the functions we have dealt with are continuous on the range of integration. Another type of improper integral arises when the integrand is discontinuous over the range of integration, due to the presence of vertical asymptotes. For example, the following integrals are improper for this very reason:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx \qquad \int_0^\pi \tan x\, dx \qquad \int_0^1 \frac{1}{x}\, dx$$

and there are many other examples. To be specific, let's consider

$$\int_{-1}^{1} \frac{1}{x}\, dx.$$

This may look like a proper integral, but it is not because $1/x$ is not defined when $x = 0$. In this situation, we have to isolate the point $x = 0$, since $x = 0$ is a vertical asymptote of the graph of $y = 1/x$, and split the integral around this point:

$$\int_{-1}^{1} \frac{1}{x}\, dx = \int_{-1}^{0} \frac{1}{x}\, dx + \int_{0}^{1} \frac{1}{x}\, dx.$$

This is still not quite right, because $1/x$ is not defined at zero, so strictly if the integral is finite then we should write

$$\int_{-1}^{1} \frac{1}{x}\, dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x}\, dx + \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x}\, dx.$$

The meaning of $a \to 0^+$ is that a approaches zero from the right (so this is a one-sided limit on the interval $[a, 1]$), and similarly $b \to 0^-$ means b approaches zero from the left. So the last two integrals are then worked out as one-sided limits. Unfortunately,

$$\lim_{a\to 0^+} \int_{a}^{1} \frac{1}{x}\, dx = \lim_{a\to 0^+} (\ln 1 - \ln|a|) = -\infty.$$

The same thing holds for the other limit as $b \to 0^-$. In this situation we say that the integral diverges. The key point to remember in this example is that we have to split the integral using the asymptotes of the function being integrated, and then take one sided limits.

It is not always the case that if the integrand is not defined on the range of integration, then the integral does not exist. For example,

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} (2 - 2\sqrt{a}) = 2.$$

So in this case the integral is finite, and we say the integral converges. In general, given an integral

$$\int_a^b f(x) dx$$

we have to seek vertical asymptotes of $f(x)$ on the interval $[a, b]$, and then split the range of integration. So if $x = c$ is an asymptote where $c \in [a, b]$, then the integral is really

$$\int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

We repeat this procedure for each of these two integrals if f has more vertical asymptotes on $[a, b]$.

12.4 Examples.

Example: We consider a second example. Suppose we wish to evaluate

$$\int_0^\pi \tan x\, dx.$$

Since $\tan x$ has an asymptote at $x = \pi/2$, we have

$$\int_0^\pi \tan x\, dx = \int_0^{\pi/2} \tan x\, dx + \int_{\pi/2}^\pi \tan x; dx$$

if the integral is finite. Since

$$\int \tan x\, dx = \ln|\sec x| + c$$

we have

$$\int_0^{\pi/2} \tan x\, dx = \lim_{b \to (\pi/2)^-} \tan b = \infty.$$

Similarly

$$\int_{\pi/2}^\pi \tan x\, dx = \tan \pi - \lim_{a \to (\pi/2)^+} \tan b = -\infty.$$

So the integral diverges. Note that we cannot "cancel" ∞ and $-\infty$ when we do this integral.

Example: Our third example is the integral

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx.$$

The function $f(x) = 1/\sqrt{1-x^2}$ has vertical asymptotes at $x = 1$ and $x = -1$. Therefore

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{a\to -1^+} \int_{a}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{a\to -1^+} \lim_{b\to 1^-} \int_{a}^{b} \frac{1}{\sqrt{1-x^2}}\, dx.$$

Since an antiderivative of the integrand is $\arcsin x$, we have

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{b\to 1^-} \arcsin b - \lim_{a\to -1^+} \arcsin a$$

Using the first fundamental theorem of calculus. Since $\arcsin x$ is continuous for $-1 \leq x \leq 1$ and $\arcsin 1 = \pi/2$ and $\arcsin(-1) = -\pi/2$, the integral equals π.

Example: Finally, let's determine the values of r for which the integral

$$\int_{0}^{1} x^r\, dx$$

converges. If $r \neq -1$, we know that

$$\int x^r\, dx = \frac{x^{r+1}}{r+1} + c.$$

The integrand is not defined at $x = 0$ if $r < 0$, so for $r < 0$ we have

$$\int x^r\, dx = \frac{1}{r+1} - \lim_{a\to 0^+} \frac{a^{r+1}}{r+1}.$$

Provided $r > -1$, the limit is zero and the integral equals $1/(r+1)$. So for $r > -1$, the integral converges. For $r < -1$, the integral diverges since the limit above is infinite. Finally, if $r = -1$, then

$$\int_{0}^{1} x^r\, dx = \ln 1 - \lim_{a\to 0^+} \ln|a|.$$

The limit is $-\infty$, so for $r = -1$ the integral diverges. We conclude that

$$\int_{0}^{1} x^r\, dx \quad \text{converges if and only if } r > -1$$

This completes the example.

Next we will see how to decide whether an integral converges without having to evaluate the integral, using comparisons to known convergent or divergent integrals.