14.1 Comparison Test

Although we have seen that it is easy to tell that the integrals
\[ \int_1^\infty \frac{1}{x} \, dx \quad \text{and} \quad \int_1^\infty \frac{1}{x^2} \, dx \]
diverge and converge, respectively, it is often not possible to do this by evaluating the integral explicitly. In the cases above, the antiderivatives were \( \ln|x| \) and \( 1/x \) so it was easy to see by taking limits that the first diverges and the second converges. However, what about the integral
\[ \int_1^\infty \frac{1}{x^2 + |\sin x|} \, dx \]
where there is little hope of finding an antiderivative? Fortunately, we can compare this integral to a known integral to see whether it diverges or not. Observe that for \( 1 \leq x \leq \infty \), it is definitely true that
\[ 0 \leq \frac{1}{x^2 + |\sin x|} \leq \frac{1}{x^2} . \]
So the integral really represents the area above the \( x \)-axis under the curve \( y = 1/(x^2 + |\sin x|) \). Since the curve is always at most \( 1/x^2 \) for \( 1 \leq x \leq \infty \), the area under it is at most the area under \( 1/x^2 \). A picture is shown below:

Curves \( y = \frac{1}{x^2 + |\sin x|} \) and \( y = \frac{1}{x^2} \)
But the integral of $1/x^2$ is finite (we said above that it converges – convergence for us means the area is finite), so that means that the integral we want also converges:

$$0 \leq \int_1^\infty \frac{1}{x^2 + |\sin x|} \, dx \leq \int_1^\infty \frac{1}{x} \, dx = 1.$$ 

So the area under the curve $y = 1/(x^2 + |\sin x|)$ is between zero and one, and the integral converges.

The main idea of the comparison test is to compare a complicated integral to a simple integral whose convergence is known. In the above example, we compared $1/(x^2 + |\sin x|)$ to $1/x^2$. Intuitively, when $x$ is very large, it should be clear that $1/(x^2 + |\sin x|)$ is very close to $1/x^2$, and this explains why $1/x^2$ is a good choice for comparison. The comparison test is as follows:

**Comparison Test**

1. Let $f(x)$ and $g(x)$ be continuous functions such that $0 \leq f(x) \leq g(x)$ for $a \leq x \leq b$. If $\int_a^b g(x) \, dx$ converges, then $\int_a^b f(x) \, dx$ converges.

2. Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x) \geq 0$ for $a \leq x \leq b$. If $\int_a^b g(x) \, dx$ diverges, then $\int_a^b f(x) \, dx$ diverges.

Recall, for our purpose, an integral diverges if it is infinite (it may by $+\infty$ or $-\infty$). So the comparison test in terms of areas should be intuitively clear: if the area under $g$ is finite, and $f \leq g$, then the area under $f$ must also be finite. If the area under $g$ is infinite and $f \geq g$, then the area under $f$ must also be infinite. In the above example, we had $f(x) = 1/(x^2 + |\sin x|)$ and $g(x) = 1/x^2$, and we used part (1) of the comparison test, since $0 \leq f(x) \leq g(x)$.

Let’s consider an example of an integral which diverges, using part (2) of the comparison test. We claim that

$$\int_1^\infty \frac{1}{x - |\sin x|} \, dx$$

diverges. The integral clearly suggests that we should compare $1/(x - |\sin x|)$ to $1/x$, because we know that the integral of $1/x$ diverges. In fact,

$$\frac{1}{x - |\sin x|} \geq \frac{1}{x} \geq 0$$

and so part (2) of the test applies:

$$\int_1^\infty \frac{1}{x - |\sin x|} \, dx \geq \int_1^\infty \frac{1}{x} \, dx = \infty.$$
and so the integral diverges.

14.2 Further Examples

Example 1.
Determine whether
\[ \int_1^\infty \frac{1}{\sqrt{x^3 + 1}} \, dx \]
diverges or converges. It is less obvious to see what to compare $1/\sqrt{x^3 + 1}$ to. However, we notice that if $x$ is large, then $1/\sqrt{x^3 + 1}$ should be very close to $1/\sqrt{x^3} = x^{-3/2}$. Since (check this)
\[ \int_1^\infty x^{-3/2} \, dx = 2 \]
this integral converges. By the comparison test part (1), with $f(x) = 1/\sqrt{x^3 + 1}$ and $g(x) = 1/\sqrt{x^3}$, the given integral converges because $0 \leq f(x) \leq g(x)$.

Example 2.
Determine whether
\[ \int_2^\infty \frac{1}{\sqrt{x^3 - 1}} \, dx \]
diverges or converges. Well again we would guess that $1/\sqrt{x^3 - 1}$ is very close to $1/\sqrt{x^3}$ when $x$ is large, and so the integral should converge. The problem is that $1/\sqrt{x^3 - 1} > 1/x^3$ so we can’t immediately apply (1) of the comparison test. To apply (1), we just notice that
\[ \frac{1}{\sqrt{x^3 - 1}} \leq \frac{2}{\sqrt{x^3}} \]
and the integral of $2/x^3$ converges, so we are done. But how do we know that this inequality is true? Well the inequality is true if and only if
\[ x^3 \leq 2(x^3 - 1) \]
for $2 \leq x < \infty$. This is the same as saying $x^3 \geq 2$ for $2 \leq x \leq \infty$, and since this is obviously true, the inequality holds.

Example 3.
Determine whether
\[ \int_4^\infty \frac{1}{x \ln x - 1} \, dx \]
converges or diverges. This is a bit trickier, but actually if we remember that
\[ \int \frac{1}{x \ln x} \, dx = \ln |\ln x| + c \]
then we can compare the given integral to
\[ \int_{4}^{\infty} \frac{1}{x \ln x} \, dx \]
which diverges (because \(\lim_{b \to \infty} \ln |\ln b| = \infty\). Clearly
\[ \frac{1}{x \ln x - 1} \geq \frac{1}{x \ln x} \geq 0 \]
when \(4 \leq x \leq \infty\), and so by the comparison test (2) this integral diverges.

**Example 4.**
Determine whether
\[ \int_{2}^{\infty} \frac{1}{x^2 + \sin x} \, dx. \]
converges or diverges. We would expect it to converge because \(\sin x\) is small compared to \(x^2\) when \(x\) is large. It makes sense to compare the integral to the integral of \(1/x^2\). The problem is that \(1/(x^2 + \sin x) \leq 1/x^2\) is not true in general, because \(\sin x\) can be negative. So we do know
\[ \frac{1}{x^2 + \sin x} \leq \frac{1}{x^2 - 1} \]
for \(2 \leq x \leq \infty\). Fortunately, \(1/(x^2 - 1) \leq 2/x^2\), because \(x^2 \leq 2(x^2 - 1)\) for \(2 \leq x \leq \infty\). Therefore
\[ \int_{2}^{\infty} \frac{1}{x^2 + \sin x} \, dx \leq \int_{2}^{\infty} \frac{2}{x^2} \, dx \]
and the integral converges.

It is a good idea to have in mind a few integrals that converge, and a few that diverge so that the comparison test can be applied. A good fact to remember is that

The integral \(\int_{1}^{\infty} x^r \, dx\) is \(\left\{ \begin{array}{ll} \text{convergent} & \text{if } r < -1 \\ \text{divergent} & \text{if } r \geq -1 \end{array} \right.\)