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Lecture 2 : Wednesday April 2nd

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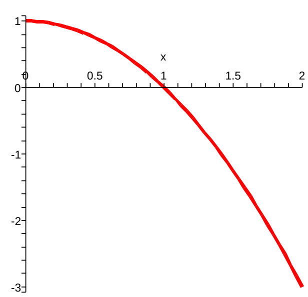
2.1 Velocity and distance

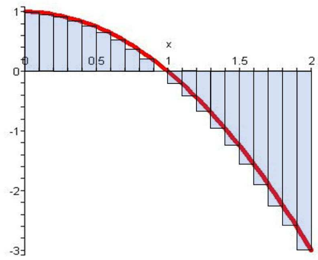
The area under a curve representing the velocity of a particle travelling in a straight line also has meaning: it is the distance that the particle has travelled over time. For example, if a particle travels in a straight line with velocity $f(t) = 1 - t^2$ feet per second for $0 \leq t \leq 1$, then the distance the particle has travelled from time $t = 0$ seconds to time $t = 1$ seconds is exactly the area between the x-axis and the curve $f(t) = 1 - t^2$ for $0 \leq t \leq 1$. Given that the area under the curve is $\frac{2}{3}$, we conclude that the origin has travelled a distance of $\frac{2}{3}$ of a foot from time $t = 0$ to time $t = 1$.

Since velocity is a quantity which also has direction, we can imagine a particle moving along a line as having positive velocity if it is moving to the right and negative velocity if it is moving to the left. If $f(t)$ describes the velocity of the particle at time t, then we can allow $f(t)$ to be negative. For example, suppose we consider the velocity to be $f(t) = 1 - t^2$ for $0 \leq t \leq 2$. Again we can ask how far the particle travelled from time $t = 0$ to time $t = 2$. A picture of the velocity curve is shown below:



Negative velocity

To work out the distance travelled by the particle from its starting point, we notice that the particle has travelled $\frac{2}{3}$ of a foot to the right in the first second, as we stated above. But then the particle starts moving to the left from $t = 1$ to $t = 2$, since the curve $1 - t^2$ dips below the t-axis. So we have to subtract the area between the curve and the t-axis for $1 \leq t \leq 2$. It turns out that the area is $\frac{4}{3}$, so this means the particle ends up $\frac{2}{3}$ of a foot to the left of its starting point when $t = 2$. The total distance travelled is therefore $\frac{2}{3}$ of a foot.



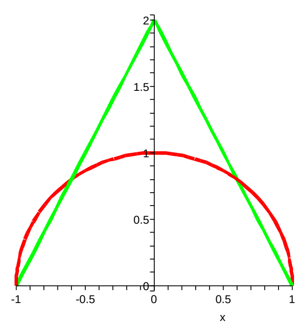
Negative velocity

The main point is that we have to count the area between the curve and the t-axis as negative when the curve dips below the t-axis to find the total distance travelled.

2.2 Area between curves

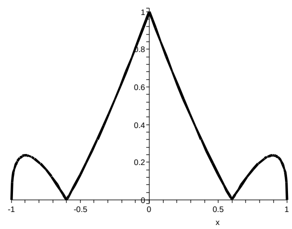
If we are just looking for the area between a curve $y = f(x)$ and the x-axis, we don't need to restrict to $f(x) \geq 0$ as we did in Lecture 1. If the curve $y = f(x)$ dips below the x-axis, we just make sure that we count that area too with a positive sign. To be precise, if we take the curve $y = 1 - x^2$ for $0 \leq x \leq 2$, then we saw in the velocity calculation that the total area between the curve and the x-axis is $\frac{2}{3} + \frac{4}{3} = 2$. There is a neater way to say this: the area between a curve $y = f(x)$ and the x-axis is the same as the area between $y = |f(x)|$ and the x-axis. Since $|f(x)| \geq 0$ for all x, we are back in the position of Lecture 1 when we try to find or approximate the area between $f(x)$ and the x-axis. Another twist is to consider the area between two curves $y = f(x)$ and $y = g(x)$. In this case, it is the same as the area between $|f(x) - g(x)|$ and the x-axis: the reason is shown in the next example.

Let's do the example of finding the area between the curves $y = 2 - 2|x|$ and $y = \sqrt{1 - x^2}$ when $0 \leq x \leq 1$. These curves are shown below.



Area between two curves

We said we have to look at the area under the curve $|f(x)-g(x)|$ to find the area between $f(x)$ and $g(x)$. So in this case we have to find the area under the curve $|2-2|x|-\sqrt{1-x^2}|$, which looks as follows:



Plot of $|f(x) - g(x)|$

The true area under this curve is actually $0.6837941090\ldots$ but let's find the approximate area as we have done before, using ten rectangles and a left Riemann sum. The rectangles therefore have base width equal to 0.2 and the heights of the rectangles in order from left to right are to two decimal places

$$0 \quad 0.2 \quad 0 \quad 0.28 \quad 0.62 \quad 1 \quad 0.62 \quad 0.28 \quad 0 \quad 0.2000000000.$$

The left Riemann sum is

$$0.2 \cdot 0.2 + 0.2 \cdot 0.28 + 0.2 \cdot 0.62 + 0.2 \cdot 1 + 0.2 \cdot 0.62 + 0.2 \cdot 0.28 + 0.2 \cdot 0.2 \approx 0.64$$

so the approximate area we get is 0.64. If you used twenty rectangles, we'd get a better approximation of 0.67.

Sums and Σ notation

In all the preceding material, we are estimating the area under curves using Riemann sums. A general left Riemann sum has the form

$$f(x_0)\Delta_0 + f(x_1)\Delta_1 + \cdots + f(x_{n-1})\Delta_{n-1}$$

when we partition an interval $[a, b]$ into subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ when $\Delta(x_i) = x_{i+1} - x_i$ is the width of the interval $[x_i, x_{i+1}]$. Similarly, a right Riemann sum has the form

$$f(x_1)\Delta_1 + f(x_2)\Delta_2 + \cdots + f(x_n)\Delta_n$$

where $\Delta_i = x_i - x_{i-1}$. We can write this in a better way using Σ notation: for a partition into n rectangles, we write the left Riemann sum as

$$\sum_{i=0}^{n-1} f(x_i)\Delta_i$$

which is read "the sum of $f(x_i)\Delta_i$ from $i = 0$ to $i = n - 1$". The Σ is the capital greek letter for the latin letter s. The right Riemann sum is

$$\sum_{i=1}^{n} f(x_i)\Delta_i.$$

These sums will be crucial in defining integration.

To get used to Σ notation, we can add up the values of any function. Instead of writing

$$f(1) + f(2) + \cdots + f(n)$$

we write

$$\sum_{i=1}^{n} f(i).$$

Here are some examples. We determine the values of the three sums

$$\sum_{i=1}^{4} i \quad \sum_{i=2}^{5} i^2 \quad \sum_{i=0}^{2} \frac{1}{i+1}.$$

The first sum written out in full is $1+2+3+4 = 10$. The second is $4+9+16+25 = 54$. The last one is $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$. You might remember also that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is the formula for the sum of the first n positive integers. We're not going to spend time in the course doing such sums, since we're going to concentrate on Riemann sums for defining integrals.