4.1 Properties of Integrals

If $f$ and $g$ are functions, then we know that for any integers $m < n$, we have

$$\sum_{i=1}^{m} f(i) + \sum_{i=m+1}^{n} f(i) = \sum_{i=1}^{n} f(i).$$

The same rule holds for integrals since they are defined by Riemann sums: for any continuous function $f$ on an interval $[a, b]$ and any real number $c \in [a, b],$

$$\int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx.$$

This makes perfect sense if we consider the case $f(x) \geq 0$: the integral on the left is the area under $f(x)$ for $a \leq x \leq b$, and this is the same as adding up the area under $f(x)$ between $a$ and $c$ and then between $c$ and $b$, as shown in the picture.

Similarly, if $f$ and $g$ are functions and $c$ is any real number, then

$$\sum_{i=1}^{n} (f(i) + g(i)) = \sum_{i=1}^{n} f(i) + \sum_{i=1}^{n} g(i)$$

$$\sum_{i=1}^{n} c \cdot f(i) = c \cdot \sum_{i=1}^{n} f(i).$$
For integrals, these become

\[
\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx
\]

\[
\int_a^b c \cdot f(x) \, dx = c \cdot \int_a^b f(x) \, dx.
\]

Together these form the property of linearity of integration. This means that if we take any linear combination \( c f(x) + d g(x) \), then

\[
\int_a^b (c f(x) + d g(x)) \, dx = c \int_a^b f(x) \, dx + d \int_a^b g(x) \, dx.
\]

Finally, perhaps the least intuitive property is

\[
\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx.
\]

So if we reverse the limits of integration, then we have to change the sign in front of the integral. The proofs of all these properties are seen from the definition of integrals via Riemann sums.

### 4.2 Examples

**Example 1.**

Suppose we want to find the area below the curve \( 1 + \frac{1}{2}x^3 \) for \( 1 \leq x \leq 2 \). Since \( 1 + \frac{1}{2}x^3 \geq 0 \) for those values of \( x \), the area is exactly

\[
\int_1^2 (1 + \frac{1}{2}x^3) \, dx.
\]

By the linearity rules of integrals, this is exactly

\[
\int_1^2 1 \, dx + \frac{1}{2} \int_1^2 x^3 \, dx.
\]

Since \( x \) is an antiderivative of 1 and \( F(x) = \frac{1}{4}x^4 \) is an antiderivative of \( x^3 \), the first integral is \( 2 - 1 = 1 \) and the second integral is

\[
\frac{1}{2} \left( F(2) - F(1) \right) = \frac{1}{2} \left( 4 - \frac{1}{4} \right) = \frac{15}{8}.
\]

So the area under \( 1 + \frac{1}{2}x^3 \) is \( \frac{23}{8} \).
Example 2.
If we want to integrate $|x|$ for $-1 \leq x \leq 1$, then we have to find an antiderivative of $|x|$. This is easy if we write the function $f(x) = |x|$ as

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The integral is then

$$\int_{1}^{1} f(x) \, dx = \int_{-1}^{0} f(x) \, dx + \int_{0}^{1} f(x) \, dx$$

$$= \int_{-1}^{0} (-x) \, dx + \int_{0}^{1} x \, dx$$

An antiderivative for $-x$ is $F(x) = -\frac{1}{2}x^2$ and an antiderivative for $x$ is $G(x) = \frac{1}{2}x^2$. By the fundamental theorem, the integral is

$$F(0) - F(-1) + G(1) - G(0) = 0 + \frac{1}{2} + \frac{1}{2} - 0 = 1.$$ 

This is what we expected because the area under $|x|$ for $-1 \leq x \leq 1$ is exactly $1$ – the region consists of two triangles whose areas are $\frac{1}{2}$ each.

The last example is important: when faced with absolute value of a function, it is wise to split the integral into components depending on where the function is positive and where the function is negative.

Example 3.
Our last example is to find the area between $\sin 2x$ and $\cos x$ for $0 \leq x \leq \pi/2$. These curves are drawn below:
The area between two curves $f$ and $g$ on $[a, b]$, from preceding lectures, is just

$$\int_a^b |f(x) - g(x)| \, dx.$$ 

In our case, we want

$$\int_0^{\pi/2} |\sin 2x - \cos x| \, dx.$$ 

Since we are integrating an absolute value, we split the integral into the parts where $\sin 2x > \cos x$ and $\sin 2x \leq \cos x$. Consistent with the picture, there is only one solution to $\sin 2x = \cos x$ for $0 \leq x \leq \pi/2$: it is when $x = \pi/6$. To see this: note that for $x < \pi/2$,

$$\sin 2x = \cos x \implies 2 \sin x \cos x = \cos x$$

$$\implies 2 \sin x = 1 \quad \text{since } \cos x \neq 0$$

$$\implies \sin x = \frac{1}{2}$$

$$\implies x = \frac{\pi}{6}.$$ 

For $x > \pi/6$, $\sin 2x > \cos x$ whereas for $x \leq \pi/6$ we have $\sin 2x \leq \cos x$. Another way of saying this is

$$|\sin 2x - \cos x| = \begin{cases} 
-\sin 2x + \cos x & \text{if } x \leq \frac{\pi}{6} \\
\sin 2x - \cos x & \text{if } x > \frac{\pi}{6}
\end{cases}$$

Therefore the integral is, by the properties of integration listed above,

$$\text{Area} = \int_0^{\pi/2} |\sin 2x - \cos x| \, dx$$

$$= \int_0^{\pi/6} (\cos x - \sin 2x) \, dx + \int_{\pi/6}^{\pi/2} (\sin 2x - \cos x) \, dx$$

$$= \int_0^{\pi/6} \cos x \, dx - \int_0^{\pi/6} \sin 2x \, dx + \int_{\pi/6}^{\pi/2} \sin 2x \, dx - \int_{\pi/6}^{\pi/2} \cos x \, dx.$$ 

To work out these four integrals, we need an antiderivative for $\sin 2x$ and for $\cos x$. Clearly $F(x) = \sin x$ is an antiderivative for $\cos x$. A good starting guess for an antiderivative for $\sin 2x$ is $\cos 2x$, except that when we take the derivative of $\cos 2x$ we get $-2 \sin 2x$. So to compensate for this, an antiderivative of $\sin 2x$ is $G(x) = -\frac{1}{2} \cos 2x$ – just check it by
taking the derivative of $G(x)$. By the fundamental theorem, the integrals are

$$
\int_{0}^{\pi/6} \cos x \, dx = F\left(\frac{\pi}{6}\right) - F(0) = \frac{1}{2}
$$
$$
\int_{0}^{\pi/6} \sin 2x \, dx = G\left(\frac{\pi}{6}\right) - G(0) = \frac{1}{4}
$$
$$
\int_{\pi/6}^{\pi/2} \sin 2x \, dx = G\left(\frac{\pi}{2}\right) - G\left(\frac{\pi}{6}\right) = \frac{3}{4}
$$
$$
\int_{\pi/6}^{\pi/2} \cos x \, dx = F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{6}\right) = \frac{1}{2}
$$

The whole integral is $\frac{1}{2} - \frac{1}{4} + \frac{3}{4} - \frac{1}{2} = \frac{1}{2}$, so the area between the two curves is $\frac{1}{2}$.