5.1 Indefinite integrals

If \( F(x) \) is an antiderivative of \( f(x) \) for \( a \leq x \leq b \), then so is \( F(x) + c \) for any constant \( c \), since the derivative of a constant function is zero. In fact, once we have an antiderivative \( F(x) \) of \( f(x) \) then every antiderivative of \( f(x) \) has the form \( F(x) + c \) for some constant \( c \).

To see why every antiderivative has the form \( F(x) + c \), suppose \( G(x) \) is another antiderivative. Then \( f(x) = F'(x) = G'(x) \) for \( a \leq x \leq b \). This means that the derivative of \( F(x) - G(x) \) is zero. Now the only function whose derivative is zero for \( a \leq x \leq b \) is the constant function, and therefore \( F(x) - G(x) = c \) for some constant \( c \) and so \( G(x) = F(x) + c \) for any antiderivative \( G(x) \) of \( f(x) \) for \( a \leq x \leq b \). The indefinite integral of a continuous function \( f \) on \([a, b]\) is defined by

\[
\int f(x) \, dx = F(x) + c.
\]

This is the family of all antiderivatives of \( f(x) \), and should not be confused with the definite integral. But as with the properties of definite integrals, linearity still holds, that is if \( f \) and \( g \) are functions and \( c \) and \( d \) are constants, then

\[
\int (c f(x) + d g(x)) \, dx = c \int f(x) \, dx + d \int g(x) \, dx.
\]

In practice we will use this rule without even saying so. A word of caution though, is that the integral of a product is not the product of the integrals in general: so do not write

false! \[
\int f(x) g(x) \, dx = \int f(x) \, dx \cdot \int g(x) \, dx
\]
false!

For example, if \( f(x) = x \) and \( g(x) = x \), then the left hand side is

\[
\int x^2 \, dx = \frac{1}{3} x^3 + c
\]
because \( \frac{1}{3} x^3 \) is an antiderivative of \( x^2 \). But \( \frac{1}{2} x^2 \) is an antiderivative of \( x \), so the right hand side is

\[
\int x \, dx \cdot \int x \, dx = \left( \frac{1}{2} x^2 \right)^2 = \frac{1}{4} x^4
\]
and this is not the same as the left hand side.

5.2 Basic Antiderivatives

If $f(x) = 0$, then we already saw above that $F(x) = c$ is an antiderivative of zero, for any constant $c$. In other words,

$$\int 0 \, dx = c.$$ 

Now if $f(x) = k$ for some $k \neq 0$, then

$$\int k \, dx = kx + c$$

since $kx$ is an antiderivative of $k$ and all antiderivatives have the form $kx + c$. More generally, if $r \geq 0$ then we see that

$$\int x^r \, dx = \frac{1}{r+1} x^{r+1} + c.$$ 

The same holds if $r$ is negative, except we do not allow $r = -1$ otherwise the right hand side is not defined. In addition we exclude $x = 0$ when $r < -1$ otherwise again the right hand side is not defined. The reason is that in this case, $\ln x$ is an antiderivative of $x^{-1}$ for $x > 0$, because

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$ 

So in this special case, provided $x > 0$, we have

$$\int \frac{1}{x} \, dx = \ln x + c.$$ 

If $x < 0$, then $\ln(-x)$ is an antiderivative of $\frac{1}{x}$. So putting these together, we have for all $x \neq 0$,

$$\int \frac{1}{x} \, dx = \ln |x| + c.$$ 

The next formula is really simple:

$$\int e^x \, dx = e^x + c$$

since $e^x$ is the derivative of $e^x$. Furthermore we also have

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$
Already we can compute quite a few integrals using these basic formulas.

5.3 Examples.

Example 1.
We find an antiderivative $F$ of $e^x + \sin x$ with the property that $F(0) = 0$. First note that $e^x - \cos x$ is an antiderivative of $e^x + \sin x$, using the rules above. So in general we have

$$\int (e^x + \sin x) \, dx = e^x - \cos x + c.$$ 

But we want this antiderivative to be zero at zero, and so

$$e^0 - \cos 0 + c = 0.$$ 

Therefore $c = 0$ and the required antiderivative is $F(x) = e^x - \cos x$.

Example 2.
We work out the integral

$$\int (\sqrt{x} + \frac{1}{\sqrt{x}}) \, dx$$ 

using the rules above. Since $\sqrt{x} = x^{1/2}$ and $1/\sqrt{x} = x^{-1/2}$,

$$\int (\sqrt{x} + \frac{1}{\sqrt{x}}) \, dx = \int x^{1/2} \, dx + \int x^{-1/2} \, dx = \frac{2}{3} x^{3/2} + 2x^{1/2} + c.$$ 

We can check the answer by making sure that its derivative is $\sqrt{x} + 1/\sqrt{x}$.

Example 3.
Let’s determine

$$\int \sin^2 x + (\pi + \cos x)^2 \, dx.$$ 

First we expand the expression

$$\sin^2 x + (\pi + \cos x)^2 = \sin^2 x + \pi^2 + 2\pi \cos x + \cos^2 x = 1 + \pi^2 + 2\pi \cos x$$ 

since $\sin^2 x + \cos^2 x = 1$. Now we integrate this to get

$$\int (1 + \pi^2 + 2\pi \cos x) \, dx = (1 + \pi^2)x + 2\pi \sin x + c.$$ 

Note that $(1 + \pi^2)$ is a constant, so an antiderivative of $(1 + \pi^2)$ is $(1 + \pi^2)x$. 

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