Lecture 7 : Monday April 14th

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7.1 Second fundamental theorem of calculus.

Try as you may, you will not be able to write down an explicit function which is an antiderivative of e^{x^2}. Nevertheless, this function does have an antiderivative. The second fundamental theorem of calculus gives antiderivatives for such functions, which can be computed numerically at any point, even though they are not explicitly written as a function of x. To explain the second fundamental theorem, first recall from the first fundamental theorem that

$$F(b) - F(a) = \int_a^b f(x)\, dx$$

when F is an antiderivative of f on $[a, b]$. We have already said that F is unique if we specify a value of F, so we might as well say $F(0) = 0$. Putting $a = 0$ and $b = t$ we get

$$F(t) = \int_0^t f(x)\, dx$$

and this defines an antiderivative of f.

Second Fundamental Theorem of Calculus
If f is a continuous function on $[a, b]$ where $a < b$, and $t \in [a, b]$, then

$$F(t) = \int_a^t f(x)\, dx$$

is an antiderivative of f.

In other words, $F'(t) = f(t)$ for $t \in [a, b]$. Let's sketch why this theorem is true when $f(x) \geq 0$ on $[a, b]$. We have to show that for any $t \in [a, b]$,

$$\lim_{h \to 0} \frac{F(t + h) - F(t)}{h} = f(t)$$

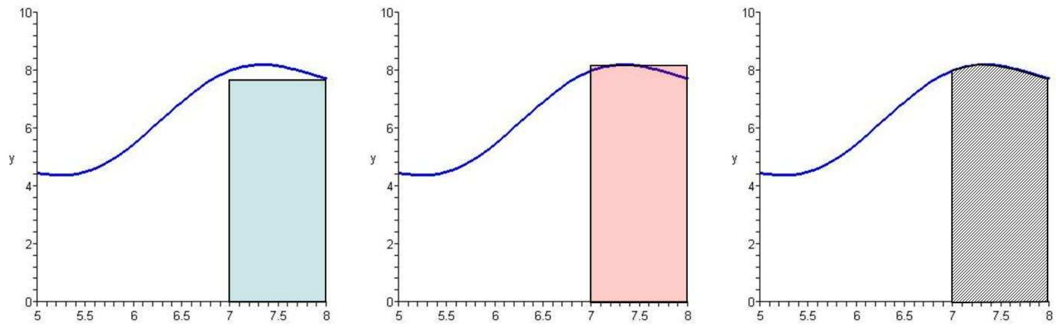
since the left hand side is exactly $F'(t)$. By the basic properties of integrals, we know

$$F(t + h) - F(t) = \int_a^{t+h} f(x)\, dx - \int_a^t f(x)\, dx = \int_t^{t+h} f(x)\, dx.$$

The last integral is the area between f and the x-axis when $t \leq x \leq t + h$. Let $\min f$ denote the smallest value of f on the interval $[t, t + h]$ and let $\max f$ denote the largest value of f on the same interval. Then comparing areas of rectangles we get

$$\min f \cdot h \leq \int_t^{t+h} f(x)\, dx \leq \max f \cdot h.$$

This is easily seen from the picture below:



Comparing areas

Therefore

$$\min f \leq \frac{F(t+h) - F(t)}{h} \leq \max f.$$

Since f is continuous,

$$\lim_{h \to 0} \min f = \lim_{h \to 0} \max f = f(t).$$

So it follows that

$$f(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} = F'(t)$$

which is what we wanted to prove.

7.2 Examples.

Example 1.
The function $f(x) = e^{-x^2}$ does not have an explicit antiderivative. By the second fundamental theorem,

$$F(t) = \int_0^t e^{-x^2}\, dx$$

is an antiderivative of e^{-x^2}. Therefore

$$\frac{d}{dt} \int_0^t e^{-x^2}\, dx = e^{-t^2}.$$

Now we can work out values of $F(t)$ by approximating the integral. Here is a table of some of the values to four decimal places:

t	1	2	3	4	5	100
$F(t)$	0.7468	0.8821	0.8862	0.8862	0.8862	0.8862

It turns out (and this is very important in probability) that

$$\lim_{t\to\infty} F(t) = \frac{1}{2}\sqrt{\pi}.$$

The function

$$\frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx$$

is often denoted erf(x), and is called the error function.

Example 2.
The function $f(x) = \frac{\sin x}{x}$ also has no explicit antiderivative. The function

$$F(t) = \int_0^t \frac{\sin x}{x}\, dx$$

is often denoted $Si(t)$ and called the Sine function. Again we know for $t > 0$ that

$$\frac{d}{dt}\int_0^t \frac{\sin x}{x}\, dx = \frac{\sin t}{t}.$$

As an exercise, compute $F(1)$, $F(2)$, $F(3)$ and so on.

Example 3.
For $t > 0$, let

$$F(t) = \int_{\sqrt{t}}^t \sin(x^2)\, dx.$$

Then we can determine $F'(t)$ as follows:

$$F(t) = \int_0^t \sin(x^2)\, dx - \int_0^{\sqrt{t}} \sin(x^2)\, dx.$$

By the second fundamental theorem,

$$\frac{d}{dt}\int_0^t \sin(x^2)\, dx = \sin t^2.$$

By the second fundamental theorem and the chain rule,

$$\frac{d}{dt}\int_0^{\sqrt{t}} \sin(x^2)\, dx = \frac{d}{dt}\sqrt{t}\cdot \sin t = \frac{\sin t}{2\sqrt{t}}.$$

Therefore

$$F'(t) = \sin t^2 + \frac{\sin t}{2\sqrt{t}}.$$

So we can work out $F'(t)$, even though $F(t)$ cannot be determined explicitly.