Lecture 9 : Friday April 18th

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9.1 Integration by parts.

By the product rule of differentiation, we know that if f and g are differentiable functions then

$$\frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x).$$

The version of this rule for antiderivatives is called integration by parts. Notice that

$$f(x)g'(x) = \frac{d}{dx}(f(x) \cdot g(x)) - f'(x)g(x)$$

and therefore

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x).$$

Let $G(x)$ be an antiderivative of a function $g(x)$. Then the rule for integration by parts is

$$\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx.$$

This can be viewed as a product rule for integration.

Example 1.
Integrate xe^x. If we let $f(x) = x$ and $g(x) = e^x$, then $G(x) = e^x$ is an antiderivative of $g(x)$ and $f'(x) = 1$. It follows that

$$\int xe^x\, dx = xe^x - \int 1 \cdot e^x dx = xe^x - e^x + c.$$

We can check that the derivative of the right hand side is xe^x, so our answer is correct.

Example 2.
Integrate $x \cos x$. In this case we let $f(x) = x$ and $g(x) = \cos x$ so that $G(x) = \sin x$ and $f'(x) = 1$. Therefore

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + c.$$

Again check that the derivative of the right hand side is $x \cos x$.

A key feature of integration by parts is to decide which function plays the role $f(x)$ and which function plays the role of $g(x)$. If we had taken $f(x) = \cos x$ and $g(x) = x$ then we would get

$$\int x \cos x\, dx = \frac{1}{2}x^2 \cos x + \frac{1}{2}\int x^2 \sin x\, dx.$$

But the second integral is even more complicated that what we started with, so this was not a profitable thing to do. A good rule in general is that if we have an integral

$$\int P(x) \cdot Q(x)\, dx$$

where $P(x)$ is a polynomial in x, then it is often (but not always) best to let $f(x) = P(x)$ and $g(x) = Q(x)$. In this way, when we use integration by parts, the integral

$$\int f'(x) G'(x)\, dx$$

is likely to be simpler because $f'(x)$ is a polynomial of lower degree than $f(x)$ (for example $f'(x) = 1$ if $f(x) = x$, $f'(x) = 2x$ if $f(x) = x^2$, and so on).

9.2 Examples.

Example 1.
Using integration by parts, we can find antiderivatives for functions which we did not have antiderivatives for previously. The first candidate is $\log x$.

$$\int \log x dx$$

gives all antiderivatives of $\log x$. To find them, let $f(x) = \log x$ and $g(x) = 1$. Then $G(x) = x$ is an antiderivative of 1 and $f'(x) = 1/x$. Using integration by parts,

$$\int \log x\, dx = x \log x - \int x \cdot \frac{1}{x}\, dx = x \log x - \int 1 dx = x \log x - x + c.$$

This is a handy formula to remember. So an antiderivative for $\log x$ is $x \log x - x$, and this can be checked by taking derivatives.

Example 2.
Here is another function: $(\cos x)^2$. Let $f(x) = \cos x$ and $g(x) = \cos x$. Then $G(x) = \sin x$ is an antiderivative for $g(x)$ and $f'(x) = -\sin x$. So

$$\int (\cos x)^2\, dx = \cos x \sin x + \int (\sin x)^2\, dx.$$

However $(\sin x)^2 = 1 - (\cos x)^2$ so

$$\int (\cos x)^2\, dx = \cos x \sin x + \int (1 - (\cos x)^2)\, dx = \cos x \sin x + x - \int (\cos x)^2\, dx.$$

If I is the integral we want, then we get

$$I = \cos x \sin x + x - I$$

and solving for I we get

$$I = \int (\cos x)^2 \, dx = \frac{1}{2} \cos x \sin x + \frac{1}{2} x + c.$$

Taking derivatives we check that this is the correct answer.

The last example was especially important because we used integration by parts to find an equation involving the integral I that we originally wanted, and then we can solve for I. Here is another example:

Example 3.
Integrate $e^x(\sin x)$. Well if $f(x) = \sin x$ and $g(x) = e^x$, then we get from integration by parts that

$$\int e^x \sin x \, dx = \sin x e^x - \int \cos x e^x \, dx.$$

This looks as though we haven't simplified anything. However if we now continue with integration by parts, letting $f(x) = \cos x$ and $g(x) = e^x$, we get

$$\int e^x \sin x \, dx = \sin x e^x - \cos x e^x - \int \sin x e^x \, dx$$

If I is the integral on the left, then

$$I = \sin x e^x - \cos x e^x - I$$

and we conclude that

$$I = \frac{1}{2}(\sin x - \cos x) e^x.$$

This is the final answer.

Example 4.
We now do an example which combines integration by parts with the substitution rule. Evaluate

$$\int x\sqrt{1-x} \, dx.$$

If $f(x) = x$ and $g(x) = \sqrt{1-x}$, then we need an antiderivative for $g(x)$. Well, if $u = \sqrt{1-x}$, this is

$$G(x) = \int \sqrt{1-x} \, dx = -\int \sqrt{u} \, du = -\frac{2}{3} u^{3/2} + c = -\frac{2}{3}(1-x)^{3/2} + c.$$

So $G(x) = -\frac{2}{3}(1-x)^{3/2}$ is an antiderivative for $g(x)$. Therefore, using integration by parts, we get

$$\int x\sqrt{1-x} \, dx = -\frac{2x}{3}(1-x)^{3/2} + \frac{2}{3} \int (1-x)^{3/2} \, dx.$$

For the second integral we use the substitution rule: if $u = 1-x$ then the same procedure as before gives

$$\int (1-x)^{3/2}\, dx = -\frac{2}{5}(1-x)^{5/2} + c.$$

It follows that

$$\int x\sqrt{1-x}\, dx = -\frac{2x}{3}(1-x)^{3/2} - \frac{4}{15}(1-x)^{5/2} + c.$$

Note that if the integral had been

$$\int x\sqrt{1-x^2}\, dx$$

then we would have directly used the substitution rule with $u = 1 - x^2$, and there would be no integration by parts.