Cycles in Planar Graphs

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Abstract

Let $\mathsf{C}(G)$ denote the set of lengths of cycles in a graph G, and let $\mathsf{CP}(n)$ denote the number of distinct subsets $\mathsf{C}(G) \subset \{1, 2, \dots, n\}$ where G is a hamiltonian planar graph on n vertices. In this paper, we prove that

$$\mathsf{CP}(n) \leq 2^{cn}$$

where $c < 1$ is an absolute positive constant. This compares with the constructive lower bound $\mathsf{CP}(n) \geq 2^{n/2}$ given by Faudree.

1 Introduction

For a graph G, we write $\mathsf{C}(G)$ for the set of lengths of cycles in G, and any subset of $\{1, 2, \dots, n\}$ which can be realized as $\mathsf{C}(G)$ for some graph G on n vertices is called a cycle set. There has been a great deal of research on the arithmetic properties of cycle sets of graphs relative to their edge-density. Early results include Bondy's Theorem [2], which states that $\mathsf{C}(G) = \{3, 4, \dots, n\}$ when the minimum degree of G is greater than $\frac{n}{2}$ the number of vertices of G, and a recent theorem of Gould, Haxell and Scott [5], which shows in particular that if G is a hamiltonian graph whose minimum degree is linear in the number of vertices of G, then $\mathsf{C}(G)$ is essentially an interval. In contrast, very little is known for sparse graphs – graphs whose degrees are not linear with respect to the number of vertices in the graph – and there are many open questions which remain. For example, Erdős and Gyárfás asked whether every graph of minimim degree at least three has a cycle of length a power of two, and this remains open. Perhaps one of the most noteworthy questions is whether $\mathsf{C}(G)$ consists of all even integers in $\{1, 2, \dots, n\}$ when G is a hamiltonian n-vertex graph of minimum degree at least $c\sqrt{n}$ and c is a sufficiently large absolute constant. The proof of such a statement seems quite out of reach at present.

In this paper, we consider the number of distinct sets $\mathsf{C}(G)$ when G is a hamiltonian planar graph. In [9], it was proved that very few subsets of $\{1, 2, \dots, n\}$ are the set of cycle lengths of a graph on n vertices, proving a conjecture of Erdős [4]. We believe that if $\mathsf{C}(n)$ is the number of distinct cycle sets in $\{1, 2, \dots, n\}$, then

$$\lim_{n \to \infty} \mathsf{C}(n)^{\frac{1}{n}} = \sqrt{2}.$$

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A constructive lower bound of Faudree, consisting of planar graphs, shows that $\mathsf{C}(n) \geq 2^{(n-1)/2}$ for all n, which supports the limiting behaviour given above. As a first step, we prove the following theorem for planar graphs:

Theorem 1 *Let $\mathsf{CP}(n)$ denote the number of cycle sets of hamiltonian planar graphs on n vertices. Then there exists a constant $c < 1$ such that*

$$\mathsf{CP}(n) \leq 2^{cn}.$$

For large n one can take $c = \frac{499}{500}$. It would be interesting to see if $\lim \mathsf{CP}(n)^{1/n}$ differs from $\lim \mathsf{C}(n)^{1/n}$, as the known constructive lower bounds on $\mathsf{C}(n)$ are all planar. We have not spent much effort on optimizing the value of the constant c coming from our proofs. It will be quite clear, however, that the techniques we use cannot give the optimal value $c = \frac{1}{2}$ without some new ideas. In fact, to indicate just how far we are from extending the theorem to all graphs, we raise the following conjecture:

Conjecture. *Let G be a hamiltonian graph of minimum degree at least three. Then there is an absolute positive constant a such that $|\mathsf{C}(G)| \geq an$.*

The techniques used to prove Theorem 1 show that the conjecture is true for planar graphs, and one can take $a = \frac{1}{4}$ in that case. Finally we use the following notation: thoughout this paper, G denotes a finite graph without loops and multiple edges, unless specifically stated otherwise, and $V(G)$ and $E(G)$ denote its vertex and edge set respectively. We write $\mathsf{C}(G)$ for the set of lengths of cycles in a graph G.

2 Induced trees and cycles

In this section, we make use duality in planar graphs to prove a variety of statements on the cycle set of a hamiltonian planar graph. Throughout this section, G is a hamiltonian plane graph with hamiltonian cycle C. We denote by G^* the dual graph of G.

Lemma 2.1 *Let E be the set of edges of G in the interior of C. Then the set of edges of G^* which are dual to the edges of E span a tree in G^*, and this tree is an induced subtree of G^* with $|E|+1$ vertices.*

Proof. It is well-known that if D is an induced cycle in G^*, then the set of edges dual to $E(D)$ form a minimal edge cut in G. Since no subset of E is an edge-cut in G, it follows that the set of edges of G^* which are dual to the edges of E span an induced forest in G^*. The number of edges in this forest is $|E|$ and the number of vertices in the forest equals the number of faces of G in the interior of C, namely $|E|+1$. This implies that the forest is a tree. ∎

Lemma 2.2 *Let T be any induced subtree of G^* and suppose $G^* - V(T)$ is connected. Let E_T^* be the set of edges of G^* which are incident with exactly one vertex of T. Then the set of edges of G which are dual to the edges of E_T^* span a cycle of length $|E_T^*|$ in G.*

Proof. The set E_T^* is an edge cut in G^*, and since T is a tree and $G^* - V(T)$ is connected, E_T^* is a minimal edge cut. Therefore the set of edges of G which are dual to E_T^* span a cycle of length exactly $|E_T^*|$ in G. ∎

We now combine the above lemmas to obtain information on the set $\mathsf{C}(G)$. This is the main lemma of this section. For an induced tree $U \subset G^*$, let ∂U denote the set of edges of G^* incident with exactly one vertex of U, and if U is a subtree of T, we write $U < T$. The sumset of sets A and B is $A + B = \{a + b : a \in A, b \in B\}$.

Lemma 2.3 *Let G be a hamiltonian planar graph with m edges and n vertices. Then there exists an induced tree $T \subset G^*$ with at least $\frac{1}{2}(m-n)+1$ vertices such that*

$$\{|\partial U| : U < T\} \subset \mathsf{C}(G).$$

Furthermore, there exist sets A and B each of size $\lceil \frac{1}{4}(m-n) \rceil$ such that $\mathsf{C}(G) \supset A + B$.

Proof. Let C be a hamiltonian cycle in G. Embed G in the plane so that C is a convex polygon, and the edges and vertices of C are the sides and vertices of the polygon, respectively. Then there are $m - n$ edges of G which are not in C, and $\frac{1}{2}(m-n)$ of these edges lie in the interior or exterior of C. Via the stereographic projection, the interior and exterior of C are interchangeable, so we can assume that C has at least $\frac{1}{2}(m-n)$ chords in its interior. Let E be the set of these chords of C. By Lemma 2.1, the set of edges of G^* which are dual to the edges of E span an induced subtree T of G^* with at least $\frac{1}{2}(m-n)$ edges. Then T has $\frac{1}{2}(m-n)+1$ vertices, and since $G^* - V(U)$ is connected for all $U < T$, the cut ∂U is a minimal edge cut in G^*. By Lemma 2.2, there is a cycle of length $|\partial U|$ in G. This proves the first statement of the lemma. For the second part of the lemma, we partition the edge set of T into two subsets E_1 and E_2, each spanning a subtree of T, and such that $|E_1| \le |E_2| \le |E_1| + 1$. Let $T_1 < T$ and $T_2 < T$ be the subtrees spanned by E_1 and E_2, and let x be the unique vertex in common to both T_1 and T_2. Let $S_1 < S_2 < \cdots < S_{t_1}$ be subtrees of T_1 where $|V(S_i)| = i$ and $V(S_1) = \{x\}$ and $S_{t_1} = T_1$, and let $U_1 < U_2 < \cdots < U_{t_2}$ be subtrees of T_2, where $|V(U_i)| = i$ and $V(U_1) = \{x\}$ and $U_{t_2} = T_2$. Then, with

$$A = \{|\partial S_i| : 1 \le i \le t_1\} \quad \text{and} \quad B = \{|\partial U_i| : 1 \le i \le t_2\}.$$

we have $\mathsf{C}(G) \supset \{|\partial U| : U < T\} \supset A + B$. The sets A and B have sizes t_1 and t_2, respectively. By the choice of T_1 and T_2, we know that $t_1, t_2 \ge \lceil \frac{1}{4}(m-n) \rceil$, so A and B (or, rather, appropriate subsets of A and B) have the required sizes. This completes the proof. ∎

It would be interesting to know how many edges a hamiltonian planar graph G requires to guarantee $\mathsf{C}(G) = \{3, 4, \ldots, n\}$ (such graphs are called pancyclic – see Bondy [2]). It seems possible that every hamiltonian planar graph on n vertices with more than $2n$ edges is pancyclic. As a simple application of Lemma 2.3, which is not necessary for the proof of Theorem 1, we obtain a short proof that every hamiltonian maximal planar graph is pancyclic. In fact even more is true – even graphs which are a few edges away from maximal planar are pancyclic – but we omit the more technical details. This result was proved in a different way by Hakimi, Schmeichel and Thomassen [7].

Corollary 2.4 *Let G be a hamiltonian maximal planar graph. Then G is pancyclic.*

Proof. A maximal planar graph on n vertices has $3n-6$ edges. Appling Lemma 2.3 with $m = 3n-6$, we see that $\mathsf{C}(G)$ contains $\{|\partial U| : U < T\}$ for some tree $T \subset G^*$ with at least $n-2$ vertices. If $U_1 < U_2 < \cdots < U_{n-2}$ are subtrees of T where $|V(U_i)| = i$, then

$$|\partial U_1| < |\partial U_2| < \cdots < |\partial U_{n-2}|$$

and $\mathsf{C}(G)$ contains $n-2$ distinct integers. This implies $\mathsf{C}(G) = \{3, 4, \dots, n\}$. ∎

3 Counting sets containing large sumsets

In the last section we showed that $\mathsf{C}(G)$ contains $A+B$ where A and B are large if the number of edges in G is large. To prove Theorem 1, we will show that very few subsets of $\{1, 2, \dots, n\}$ contain a set of the form $A+B$ for some linear-size sets A and B. The aim of this section is to quantify this statement, using discrete Fourier analysis. The same method was used in [9]. A recent approach by Green and Ruzsa [6] determines the number of subsets of $\mathbb{Z}_n$ of the form $A+A$ when n is prime. It would be interesting to see if this approach carries over to counting cycle sets, although there are some difficulties to overcome, and one would need a description of the whole cycle set, not only large subsets of it. The following well-known entropy bound on binomial coefficients is needed. The bound follows fairly easily from Stirling's Formula. Let $a, b \in (0,1)$, and let $H(x) = -x \log x - (1-x)\log x$. denote the entropy of $x \in (0,1)$. Then

$$\binom{an}{bn} \leq e^{anH(\frac{b}{a})}. \tag{1}$$

In what follows we will make frequent use of this inequality. We are now ready to prove that there are few sets containing large sumsets:

Lemma 3.1 *Let $s(n)$ denote the number of subsets of $\{1, 2, \dots, n\}$ which contain a set of the form $A+B$ where $|A| = an = |B|$. Then*

$$s(n) \leq n(n-1)e^{nH(\gamma)} \tag{2}$$

whenever γ satisfies the inequality

$$H(\gamma) \geq \log 2 - 2\gamma^2 a^2. \tag{3}$$

Proof. The number of subsets of $\{1, 2, \dots, n\}$ containing a set of the form $A+B$ where $|A| = an = |B|$ is equal to the number of subsets of $\mathbb{Z}_n$ containing a set of the form $A+B$. Let $S \subset \mathbb{Z}_n$ be a set containing $A+B$ where $|A| = an = |B|$ and let $T = \mathbb{Z}_n \backslash S$. Let $\hat{T}$ denote the Fourier transform of the characteristic function χ_T of T, namely

$$\hat{T}(r) = \sum_{t=0}^{n-1} \chi_T(t) e^{\frac{2\pi i r t}{n}}.$$

We claim that if $|\hat{T}(r)|$ for $r \neq 0$ achieves a maximum value at r_0, then

$$|\hat{T}(r_0)| \geq a|T|. \tag{4}$$

To see this, note that $T \cap (A+B) = \emptyset$, or equivalently, the equation $t-x-y=0$ has no solution with $t \in T$, $x \in A$ and $y \in B$. Since the sum of the roots of unity is zero, one has

$$\sum \chi_T(t)\chi_A(x)\chi_B(y)e^{\frac{2\pi i r(t-x-y)}{n}} = 0$$

where χ_A and χ_B are the characteristic functions of A and B and the sum is over $r, t, x, y \in \mathbb{Z}_n$. When $r=0$, the summand above is $|T||A||B|$. Therefore taking absolute values and writing $\hat{A}$ and $\hat{B}$ for the transforms of χ_A and χ_B, we obtain

$$\begin{aligned}
|T||A||B| &= \left| \sum_{r\neq 0} \chi_T(t)\chi_A(x)\chi_B(y)e^{\frac{2\pi i r(t-x-y)}{n}} \right| \\
&\leq \sum_{r\neq 0} |\hat{T}(r)||\hat{A}(r)||\hat{B}(r)| \\
&\leq |\hat{T}(r_0)| \sum_{r=0}^{n-1} |\hat{A}(r)||\hat{B}(r)| \\
&\leq |\hat{T}(r_0)| \left(\sum_{r=0}^{n-1} |\hat{A}(r)|^2 \right)^{\frac{1}{2}} \left(\sum_{r=0}^{n-1} |\hat{B}(r)|^2 \right)^{\frac{1}{2}} \\
&= |\hat{T}(r_0)|(|A|n)^{\frac{1}{2}}(|B|n)^{\frac{1}{2}}.
\end{aligned}$$

In the fourth line we used the Cauchy-Schwarz Inequality, and in the fifth line we used Parseval's Identity, namely

$$\sum_{r=0}^{n-1} |\hat{A}(r)|^2 = n \sum_{r=0}^{n-1} \chi_A(r) = n|A|.$$

Now (4) follows from $|A| = an = |B|$. Suppose the elements of S are chosen uniformly and independently from $\mathbb{Z}_n$ with probability $\frac{1}{2}$. Let E be the event (4). To finish the proof of the lemma, we have to prove that

$$\mathbb{P}[E] \leq (n-1)n2^{-2\log_2 e\gamma^2 a^2 n}. \tag{5}$$

To see this, we note that $|T|$ has a binomial distribution with mean $\frac{n}{2}$, and by Hoeffding's Inequality (see Proposition 2.1 and Lemma 2.3 in [9], and McDiarmid [8] for a nice article on concentration inequalities), for $r \neq 0$,

$$\mathbb{P}[|\hat{T}(r)| \geq t] \leq e^{-2t^2/n}.$$

Therefore for any $r \neq 0$,

$$\begin{aligned}
\mathbb{P}[|\hat{T}(r)| \geq a|T|] &= \sum_{t=0}^{n-1} \mathbb{P}[|\hat{T}(r)| \geq ta \cap |T| = t] \\
&\leq \sum_{t=0}^{n-1} \min\left\{ e^{-\frac{2t^2a^2}{n}}, \binom{n}{t} 2^{-n} \right\} \\
&\leq n \sup_{\gamma \in (0,1)} \min\left\{ e^{-2\gamma^2 a^2 n}, \binom{n}{\gamma n} 2^{-n} \right\} \\
&\leq n \sup_{\gamma \in (0,1)} \min\left\{ e^{-2\gamma^2 a^2 n}, e^{nH(\gamma)} 2^{-n} \right\} \\
&\leq n e^{nH(\gamma)}.
\end{aligned}$$

In the last line we used (3). Since there are $n-1$ possible values for r_0, this gives (5). ∎

4 Counting Planar Graphs

To prove Theorem 1 we have to consider the contribution to $\mathsf{CP}(n)$ of those cycle sets of hamiltonian planar graphs which have few edges (this is quantified later on). We will prove that the number of hamiltonian planar graphs with m edges and n vertices is much less than 2^n whenever m is not too much larger than m, and this clearly gives an upper bound on the contribution of these graphs to $\mathsf{CP}(n)$. The next lemma counts the number of hamiltonian planar graphs with m edges. The problem of counting planar blocks was solved by Bender, Gao and Wormald [1] using analytic techniques. The bound we give below is weaker, however the proof is very much simpler and suffices for our purposes.

Lemma 4.1 *Let $u(m,n)$ denote the number of unlabelled hamiltonian planar graphs with m edges and n vertices, where $m > n$. Then*

$$u(m,n) \leq (m-n)e^{2mH(1-\frac{n}{m})}2^{2m-2n} \tag{6}$$

Proof. Let G be a hamiltonian planar graph. Draw a hamiltonian cycle C of G in the plane as a convex polygon, and all remaining edges of G as chords of C. Then the number of graphs

G is at most the number of ways to place chords in the interior of C times the number of ways to place chords in the exterior of C. Let G_0 denote the graph obtained by deleting all exterior chords of C.then we may contract pairs of adjacent vertices of C until we obtain a triangulation H_0 with the same number of edges as G_0. Let i denote the number of chords of C in G_0, and let C_0 denote the hamiltonian cycle of H_0 corresponding to C. Then C_0 has length $i+3$, and we may order the vertices of C_0 as $(v_1, v_2, \ldots, v_{i+3})$ in the clockwise orientation of C_0. We will show that H_0 is uniquely determined by a pair (H_0, σ), where σ is a composition of n into $2i$ parts. For a given triangulation H_0 with hamiltonian cycle C_0, the composition σ is defined as follows. First let F denote the graph obtained from H_0 by deleting all edges $\{v_h, v_{h+1}\} \in E(C_0)$ such that v_h has degree two in H_0. For each $\{v_h, v_{h+1}\} \in E(C_0) \cap E(F)$, assign a non-negative integer x_h, denoting the number of vertices inbetween v_h and v_{h+1} in C. Define

$$\tau = (x_h : \{v_h, v_{h+1}\} \in E(C_0) \cap E(F)).$$

Now we list edges incident with v_h counterclockwise around v, say $e_{1,h}, e_{2,h}, \ldots, e_{d_h,h}$. Let $y_{g,h}$ denote the number of vertices in the subpath of C_0 joining the endpoints of $e_{g,h}$ and $e_{g+1,h}$ which were contracted into v_h. Then we obtain a sequence $\sigma_h = (y_{1,h}, y_{2,h}, \ldots, y_{d_h,h})$ for each $h \geq 1$. Finally, σ is defined by

$$\sigma = \tau\sigma_1\sigma_2 \ldots \sigma_{i+3}$$

and σ has length $2i$. This is illustrated in Figure 1, with $i = 6$ and $\sigma = (0, 2, 2, 1, 2, 2, 0, 1, 0, 0, 2, 1)$. The dotted lines denote edges of C_0 which are not in F.

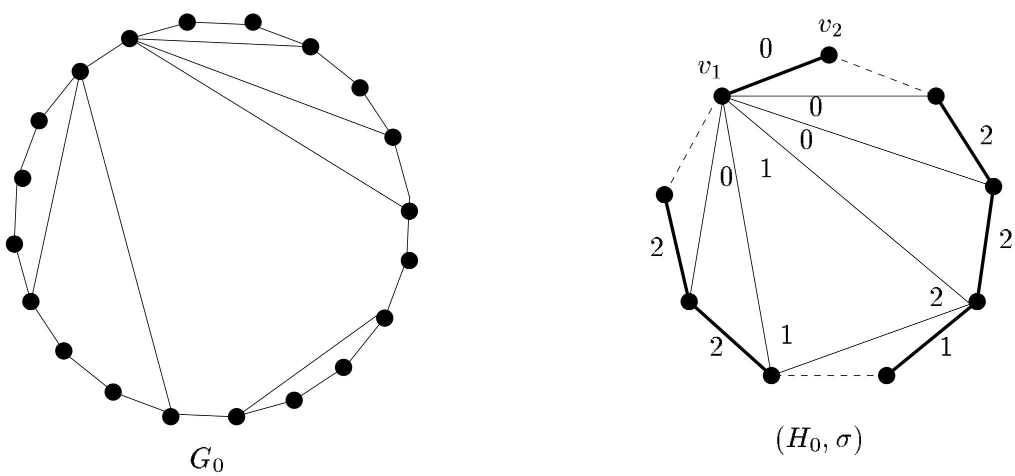

Figure 1.

Now we count the number of pairs (H_0, σ). The number of compositions of n into $2i$ non-negative parts is exactly $\binom{n+2i-1}{2i-1}$. Euler determined the number of triangulations of an r-gon, and that number is exactly the Catalan number $\frac{1}{r-1}\binom{2r-4}{r-2}$. Since C_0 is an $(i+3)$-gon, the number $f(i)$ of graphs G_0 with i edges in the interior of C, using (1), is at most

$$\binom{n+2i-1}{2i-1} \cdot \frac{1}{i+2}\binom{2i+2}{i+1} \quad < \quad \exp\left\{(n+2i)H\left(\frac{2i}{n+2i}\right)\right\} \cdot 2^{2i}.$$

The same approach gives an upper bound on the number of graphs G_1 consisting of C and all exterior chords of C. Furthermore, the number of planar graphs with i edges in the interior of C and $j = m - n - i$ edges in the exterior of C is at most the product of $f(i)$ and $f(j)$. This implies that $u(m, n)$ is at most the sum over all i and j adding up to n of $f(i)f(j)$. It follows that

$$u(m,n) \quad < \quad \sum_{\substack{i+j \\ =m-n}} \exp\left\{(n+2i)H\Big(\frac{2i}{n+2i}\Big) + (n+2j)H\Big(\frac{2j}{n+2j}\Big)\right\} \cdot 2^{2i+2j}.$$

The summand is maximized when $2i = 2j = m - n$, by symmetry, and it follows that $u(m, n)$ is at most $m - n$ times that value, giving (6). ∎

5 Proof of Theorem 1

Let $\mathcal{P}$ and $\mathcal{Q}$ denote the set of all hamiltonian planar graphs on n vertices with at most $(1+\alpha)n$ edges and more than $(1+\alpha)n$ edges, respectively, where $\alpha > 0$ is to be chosen later. According to the bound (6) on the number of graphs in $\mathcal{P}$ in Lemma 4.1, we have

$$|\{\mathsf{C}(G) : G \in \mathcal{P}\}| \;\le\; |\mathcal{P}| \;\le\; e^{2(1+\alpha)n\cdot H\left(1-\frac{1}{1+\alpha}\right)} 2^{2\alpha n}. \tag{7}$$

Next we consider $\mathcal{Q}$. For each planar graph $G \in \mathcal{Q}$, Lemma 2.3 shows that $\mathsf{C}(G)$ contains $A+B$ for some sets A and B in $\{1, 2, \ldots, n\}$ where $|A| = |B| = \lceil \frac{\alpha n}{4} \rceil$ each. By Lemma 3.1,

$$|\{\mathsf{C}(G) : G \in \mathcal{Q}\}| \quad \le \quad s(n) \quad \le \quad n(n-1)e^{nH(\gamma)} \tag{8}$$

where γ is defined by (3) with $a = \frac{\alpha}{4}$. Adding (7) and (8) gives an upper bound on $\mathsf{CP}(n)$. Finally we determine the value of α for which the logarithms of (7) and (8) are asymptotically equal as n tends to infinity. This is when

$$\frac{1}{4}\alpha^2\gamma^2 = 2\alpha \log\frac{\alpha}{\alpha+1} + 2\log\frac{1}{\alpha+1} - 2\alpha(\log 2) + \log 2.$$

Then we solve for the value of α using (3), to obtain $0.081779 < \alpha < 0.08178$. In particular

$$\mathsf{CP}(n) < e^{0.6913n} < 2^{499n/500}$$

provided n is sufficiently large. This completes the proof of Theorem 1. ∎

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