Class Quiz : Integration

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3pm - 3.50pm Center Hall 115

No notes allowed

All questions carry equal weight

Question 1.

Determine the value of

$$\int_1^2 (x + \sqrt{x} + 1)\, dx$$

Solution.
An antiderivative of $x + \sqrt{x} + 1$ is

$$F(x) = \int (x + \sqrt{x} + 1)\, dx = \frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + x + c.$$

The first fundamental theorem gives

$$\int_1^2 (x + \sqrt{x} + 1)\, dx = F(2) - F(1) = \frac{4}{3}\sqrt{2} + \frac{11}{6}.$$

Question 2.

Use the substitution $u = \sqrt{x}$ to determine

$$\int \frac{1}{x + \sqrt{x}}\, dx$$

Solution.
If $u = \sqrt{x}$ then $x = u^2$ so $\frac{dx}{du} = 2u$. Therefore by the substitution rule

$$\begin{aligned} \int \frac{1}{x + \sqrt{x}} dx &= \int \frac{2u}{u^2 + u}\, du \\ &= \int \frac{2}{u+1}\, du = 2 \ln |u + 1|. \end{aligned}$$

Question 3.

Using integration by parts, find the exact value of

$$\int_0^{\ln 2} (x+1)e^{-x}\, dx$$

Solution.
Let $F(x) = \int (x+1)e^{-x}\, dx$. Then using integration by parts, differentiating $x+1$ and integrating e^{-x}, we get

$$\begin{aligned} F(x) &= -(x+1)e^{-x} + \int e^{-x}\, dx \\ &= -(x+1)e^{-x} - e^{-x} + c \\ &= -(x+2)e^{-x} + c. \end{aligned}$$

By the first fundamental theorem,

$$\begin{aligned} \int_0^{\ln 2} (x+1)e^{-x}\, dx &= F(\ln 2) - F(0) \\ &= -(\ln 2 + 2)e^{-\ln 2} + 2 \\ &= 1 - \frac{\ln 2}{2}. \end{aligned}$$

Question 4.

Solution.
Find the exact value of

$$\int_1^{\infty} \left(\frac{\ln x}{x}\right)^2\, dx$$

Let $F(x) = \int \left(\frac{\ln x}{x}\right)^2\, dx$. We use the substitution rule with $u = \ln x$. Then $\frac{du}{dx} = \frac{1}{x}$ and $1/x = e^{-u}$. So

$$\frac{(\ln x)^2}{x^2} = u^2 e^{-u} \frac{du}{dx}.$$

It follows that

$$F(x) = \int u^2 e^{-u} du.$$

This integral is done using integration by parts. We differentiate u^2 and integrate e^{-u}:

$$F(x) = -u^2 e^{-u} + 2\int u e^{-u} du = -u^2 e^{-u} - 2ue^{-u} + 2\int e^{-u} du = -(u^2 + 2u + 2)e^{-u} + c.$$

Since $u = \ln x$,

$$F(x) = -((\ln x)^2 + 2\ln x + 2)e^{-\ln x} + c = -\frac{1}{x}((\ln x)^2 + 2(\ln x) + 2) + c.$$

Finally,

$$\int_0^\infty \left(\frac{\ln x}{x}\right)^2 \, dx = \lim_{b\to\infty} F(b) - F(1) = 2.$$