

# Note – Second derivative test

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If  $U \subseteq \mathbb{R}^n$  is a [closed and bounded set](#) and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous on  $U$ , then  $f$  has a global maximum and minimum on  $U$ . To find these using the second derivative test, proceed as follows when  $U$  has a smooth boundary:

- Determine the critical points of  $f$  by solving  $\nabla f = 0$ .
  - Discard critical points which are not contained in  $U$ .
- Determine  $f(a)$  for each critical point  $a \in U$ .
- Use the second derivative test to determine which values  $f(a)$  are local minima, maxima on  $U$ , and where the test fails.
- On the boundary  $\partial U$  of  $U$ , use the preceding steps to determine all local extremes.
- Compare all the values found to find the global minima and maxima.

The second derivative test (the third step) can fail even though  $f(a)$  is a local maximum or minimum. For example, if  $f(x, y, z) = x^4 + y^4 + z^4$ , it is clear that  $f(0, 0, 0) = 0$  is a local minimum of  $f$ , whereas the Hessian matrix is the all-zero matrix, and the second derivative test fails. The [boundary  \$\partial U\$](#)  is often described by equations relating the variables. So in the fifth step, we use these equations to eliminate some variables in  $f$  and then find the local extremes of the new function. The preferred method for working on  $\partial U$  will be the [Method of Lagrange Multipliers](#).

## An example

Find the [global minimum and maximum value](#) of the function  $f(x, y) = x^2 + y^2 + \cos x^2 + \cos y^2$  on the region  $U$  defined by  $x^2 + y^2 \leq 1$ .

**Critical points inside  $U$ .** The critical points of  $f$  are found from  $\nabla f(x, y) = 0$  and so we get the equations

$$2x - 2x \sin x^2 = 0 \quad 2y - 2y \sin y^2 = 0$$

and therefore the critical points are when  $x^2 \in \{0, k\pi + \pi/2\}$  and  $y^2 \in \{0, m\pi + \pi/2\}$  where  $k, m$  are non-negative integers. Clearly the critical point  $(0, 0)$  is contained in  $U$ , but no other critical points are contained in  $U$ .

**Second derivative test.** Now  $f(0, 0) = 2$  and  $f_{xx}(0, 0) = 2 = f_{yy}(0, 0)$  whereas  $f_{xy}(0, 0) = 0$ . Therefore the Hessian matrix is

$$H_f(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

This matrix is positive definite and so  $f$  has a local minimum at  $(0, 0)$ .

**The boundary  $\partial U$ .** The next step is to move on to the boundary of  $U$ , namely  $x^2 + y^2 = 1$ . There the function  $f$  equals

$$g(x) = 1 + \cos x^2 + \cos(1 - x^2)$$

where  $-1 \leq x \leq 1$ . We can find minima and maxima of this function using one dimensional optimization. Namely,

$$\frac{dg}{dx} = -2x \sin x^2 + 2x \sin(1 - x^2) = 0$$

for the critical points of  $g$ . Now either  $x = 0$  or

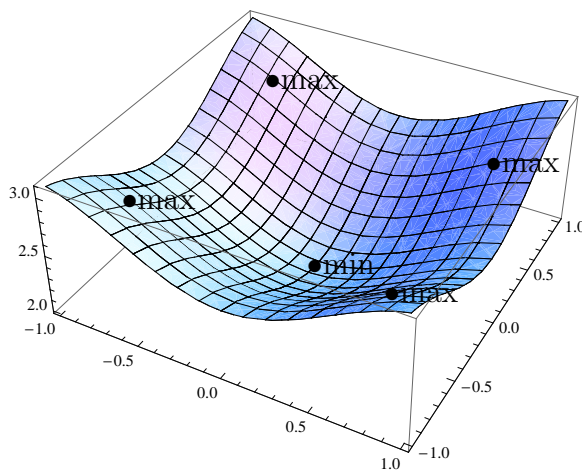
$$\sin x^2 = \sin(1 - x^2)$$

and the last equation means  $x^2 = 1 - x^2$  since  $-1 \leq x \leq 1$ . So the critical points of  $g$  are  $x = 0$  and  $x^2 = \frac{1}{2}$ . At  $x = 0$  we see that  $g$  has a minimum and at  $x^2 = \frac{1}{2}$  we see that  $g$  has a maximum, since  $\frac{d^2g}{dx^2} > 0$  when  $x = 0$  and  $\frac{d^2g}{dx^2} < 0$  when  $x^2 = \frac{1}{2}$ . Since  $g$  is optimized over  $-1 \leq x \leq 1$ , we have to consider the boundary points  $x = \pm 1$ . At these points  $g(1) = g(-1) = 1 + 2 \cos 1$ .

**Comparing values.** Now

$$g(0) = 2 + \cos 1 \quad g(\pm 1) = 1 + 2 \cos 1 \quad g\left(\pm \frac{1}{\sqrt{2}}\right) = 1 + 2 \cos \frac{1}{2}$$

and we must compare these values with  $f(0, 0) = 2$ . The first value is  $2.5403\dots$ , the second is  $2.0846$  and the third is  $2.75517\dots$ , so the final conclusion is that  $f$  has a global minimum on  $U$  at  $(0, 0)$ , and a global maximum at  $(x^2, y^2) = (\frac{1}{2}, \frac{1}{2})$ . The global minimum and maximum values on  $U$  are  $2$  and  $1 + 2 \cos \frac{1}{2}$ . The picture below supports this.



$$f(x, y) = x^2 + y^2 + \cos x^2 + \cos y^2$$