Turán problems and shadows III: expansions of graphs

Alexandr Kostochka∗ Dhruv Mubayi† Jacques Verstraëte‡

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Abstract

The expansion $G^+$ of a graph $G$ is the 3-uniform hypergraph obtained from $G$ by enlarging each edge of $G$ with a new vertex disjoint from $V(G)$ such that distinct edges are enlarged by distinct vertices. Let $\text{ex}_3(n,F)$ denote the maximum number of edges in a 3-uniform hypergraph with $n$ vertices not containing any copy of a 3-uniform hypergraph $F$. The study of $\text{ex}_3(n,G^+)$ includes some well-researched problems, including the case that $F$ consists of $k$ disjoint edges [6], $G$ is a triangle [5, 9, 18], $G$ is a path or cycle [12, 13], and $G$ is a tree [7, 8, 10, 11, 14]. In this paper we initiate a broader study of the behavior of $\text{ex}_3(n,G^+)$. Specifically, we show

$$\text{ex}_3(n,K_{t,t}^+) = \Theta(n^{3-3/s})$$

whenever $t > (s-1)!$ and $s \geq 3$. One of the main open problems is to determine for which graphs $G$ the quantity $\text{ex}_3(n,G^+)$ is quadratic in $n$. We show that this occurs when $G$ is any bipartite graph with Turán number $o(n^2)$ where $\varphi = \frac{1+\sqrt{5}}{2}$, and in particular, this shows $\text{ex}_3(n,G^+) = O(n^2)$ when $G$ is the three-dimensional cube graph.

1 Introduction

An $r$-uniform hypergraph $F$, or simply $r$-graph, is a family of $r$-element subsets of a finite set. We associate an $r$-graph $F$ with its edge set and call its vertex set $V(F)$. Given an $r$-graph $F$, let $\text{ex}_r(n,F)$ denote the maximum number of edges in an $r$-graph on $n$ vertices that does not contain $F$. The expansion of a graph $G$ is the 3-graph $G^+$ with edge set $\{e \cup \{v_e\} : e \in G\}$ where $v_e$ are distinct vertices not in $V(G)$. By definition, the expansion of $G$ has exactly $|G|$ edges. Note that Füredi and Jiang [10, 11] used a notion of expansion to $r$-graphs for general $r$, but this paper considers only 3-graphs.

Expansions include many important hypergraphs who extremal functions have been investigated, for instance the celebrated Erdős-Ko-Rado Theorem [6] for 3-graphs is the case of expansions of a matching. A well-known result is that $\text{ex}_3(n,K_3^+) = \binom{n-1}{2}$ [5, 9, 18]. If a graph is not 3-colorable

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∗University of Illinois at Urbana–Champaign, Urbana, IL 61801 and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia. E-mail: kostochk@math.uiuc.edu. Research of this author is supported in part by NSF grant DMS-1266016 and by grants 12-01-00631 and 12-01-00448 of the Russian Foundation for Basic Research.

†Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607. E-mail: mubayi@uic.edu. Research partially supported by NSF grants DMS-0969092 and DMS-1300138.

‡Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112, USA. E-mail: jverstra@math.ucsd.edu. Research supported by NSF Grant DMS-1101489.
then its expansion has positive Turán density and this case is fairly well understood [16, 19], so we focus on the case of expansions of 3-colorable graphs. It is easy to see that $\text{ex}_3(n, G^+) = \Omega(n^2)$ unless $G$ is a star (the case that $G$ is a star is interesting in itself, and for $G = P_2$ determining $\text{ex}_3(n, G^+)$ constituted a conjecture of Erdős and Sós [7] which was solved by Frankl [8]). The authors [13] had previously determined $\text{ex}_3(n, G^+) = \Omega(n^2)$ when $G$ is a forest asymptotically in [14], thus settling a conjecture of Füredi [10]. The following straightforward result provides general bounds for $\text{ex}_3(n, G^+)$ in terms of the number of edges of $G$.

**Proposition 1.1.** If $G$ is any graph with $v$ vertices and $f \geq 4$ edges, then for some $a > 0$,

$$an^{3-\frac{3v-9}{f-3}} \leq \text{ex}_3(n, G^+) \leq (n-1)\text{ex}_2(n, G) + (f + v - 1)\binom{n}{2}.$$

The proof of Proposition 1.1 is given in Section 3. Some key remarks are that $\text{ex}_3(n, G^+)$ is not quadratic in $n$ if $f > 3v - 6$, and if $G$ is not bipartite then the upper bound in Proposition 1.1 is cubic in $n$. This suggests the question of identifying the graphs $G$ for which $\text{ex}_3(n, G^+) = O(n^2)$, and in particular evaluation of $\text{ex}_3(n, G^+)$ for planar $G$.

### 1.1 Expansions of planar graphs

We give a straightforward proof of the following proposition, which is a special case of a more general result of Füredi [10] for a larger class of triple systems.

**Proposition 1.2.** Let $G$ be a graph with treewidth at most two. Then $\text{ex}_3(n, G^+) = O(n^2)$.

On the other hand, there are 3-colorable planar graphs $G$ for which $\text{ex}_3(n, G^+)$ is not quadratic in $n$. To state this result, we need a definition. A proper $k$-coloring $\chi : V(G) \to \{1, \ldots, k\}$ is acyclic if every pair of color classes induces a forest in $G$. We pose the following question:

**Question 1.** Does every planar graph $G$ with an acyclic 3-coloring have $\text{ex}_3(n, G^+) = O(n^2)$?

Let $g(n, k)$ denote the maximum number of edges in an $n$-vertex graph of girth larger than $k$.

**Proposition 1.3.** Let $G$ be a planar graph such that in every proper 3-coloring of $G$, every pair of color classes induces a subgraph containing a cycle of length at most $k$. Then $\text{ex}_3(n, G^+) = \Omega(n^2 + \Theta(\frac{1}{k}))$.

The last statement follows from the known fact that $g(n, k) \geq n^{1+\Theta(\frac{1}{k})}$. The octahedron graph $O$ is an example of a planar graph where in every proper 3-coloring, each pair of color classes induces a cycle of length four, and so $\text{ex}_3(n, O^+) = \Omega(n^{5/2})$. Even wheels do not have acyclic 3-colorings, and we do not know whether their expansions have quadratic Turán numbers.

**Question 2.** Does every even wheel $G$ have $\text{ex}_3(n, G^+) = O(n^2)$?
1.2 Expansions of bipartite graphs

The behavior of $\text{ex}_3(n, G^\ast)$ when $G$ is a dense bipartite graph is somewhat related to the behavior of $\text{ex}_2(n, G)$ according to Proposition 1.1. In particular, Proposition 1.1 shows for $t \geq s \geq 2$ and some constants $a, c > 0$ that

$$an^{3-\frac{3+3t-9}{s+t-4}} \leq \text{ex}_3(n, K_{s,t}^+) \leq cn^{3-\frac{1}{s+t-4}}.$$  

We show that both the upper and lower bound can be improved to determine the order of magnitude of $\text{ex}_3(n, K_{s,t}^+)$ when good constructions of $K_{s,t}$-free graphs are available (see Alon, Rónyai and Szabo [2]):

**Theorem 1.4.** Fix $3 \leq s \leq t$. Then $\text{ex}_3(n, K_{s,t}^+) = O(n^{3-\frac{3}{s}})$ and, if $t > (s-1)! \geq 2$, then $\text{ex}_3(n, K_{s,t}^+) = \Theta(n^{3-\frac{3}{s}})$.

The following closely related problem was recently investigated by Alon and Shikhelman [3]. For a graph $F$, let $g(n, F)$ denote the maximum number of triangles in an $n$-vertex graph that contains no copy of $F$ as a subgraph. From a graph $G$ achieving this maximum, we can form a 3-graph $H$ with $V(H) = V(G)$ and $H$ consists of the triangles in $G$. Then a copy $K$ of $F^+$ in $H$ would yield a copy of $F$ in $G$ as $\partial K \supset F$. Consequently, we have

$$g(n, F) \leq \text{ex}_3(n, F^+).$$

Alon and Shikhelman [3] independently proved that for fixed $3 \leq s \leq t$ and $t > (s-1)!$ we have $g(n, K_{s,t}) = \Theta(n^{3-3/s})$. Their lower bound construction is exactly the same as ours, though the proofs are different.

The case of $K_{3,t}$ is interesting since $\text{ex}_3(n, K_{3,t}^+) = O(n^2)$, and perhaps it is possible to determine a constant $c$ such that $\text{ex}_3(n, K_{3,3}^+) ~ cn^2$, since the asymptotic behavior of $\text{ex}_2(n, K_{3,3})$ is known, due to a construction of Brown [4] and the upper bounds of Füredi [10]. In general, the following bounds hold for expansions of $K_{3,t}$:

**Theorem 1.5.** For fixed $r \geq 1$ and $t = 2r^2 + 1$, we have $(1-o(1))\frac{t-1}{12}n^2 \leq \text{ex}_3(n, K_{3,t}^+) = O(n^2)$.

The upper bound in this theorem is a special case of a general upper bound for all graphs $G$ with $\sigma(G^\ast) = 3$ (see Theorem 1.7). Finally, we prove a general result that applies to expansions of a large class of bipartite graphs.

**Theorem 1.6.** Let $G$ be a graph with $\text{ex}_2(n, G) = o(n^\varphi)$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. Then $\text{ex}_3(n, G^\ast) = O(n^2)$.

Let $Q$ be the graph of the 3-dimensional cube (with 8 vertices and 12 edges). Erdős and Simonovits [7] proved $\text{ex}_2(n, Q) = O(n^{1.6}) = o(n^\varphi)$, so a corollary to Theorem 1.6 is that

$$\text{ex}_3(n, Q^+) = \Theta(n^2).$$

Determining the growth rate of $\text{ex}_2(n, Q)$ is a longstanding open problem. Since it is known that for any graph $G$ the 1-subdivision of $G$ has Turán Number $O(n^{3/2})$ – see Alon, Krivelevich...
and Sudakov [1] – Theorem 1.6 also shows that for such graphs $G$, $\text{ex}_3(n, G^+) = \Theta(n^2)$. Erdős conjectured that $\text{ex}_2(n, G) = O(n^{3/2})$ for each 2-degenerate bipartite graph $G$. If this conjecture is true, then by Theorem 1.6, $\text{ex}_3(n, G^+) = O(n^2)$ for any 2-degenerate bipartite graph $G$.

1.3 Crosscuts

A set of vertices in a hypergraph containing exactly one vertex from every edge of a hypergraph is called a **crosscut** of the hypergraph, following Frankl and Füredi [9]. For a 3-uniform hypergraph $F$, let $\sigma(F)$ be the minimum size of a crosscut of $F$ if it exists, i.e.,

$$\sigma(F) := \min\{|X| : \forall e \in F, |e \cap X| = 1\}$$

if such an $X$ exists. Since the triple system consisting of all edges containing exactly one vertex from a set of size $\sigma(F) - 1$ does not contain $F$, we have

$$\text{ex}_3(n, F) \geq (\sigma(F) - 1 + o(1))\left(\begin{array}{c}n \\ 2 \end{array}\right). \tag{1}$$

An intriguing open question is: For which $F$ an asymptotic equality is attained in (1)? Recall that a graph has tree-width at most two if and only if it has no subdivision of $K_4$. Informally, these are subgraphs of a planar graph obtained by starting with a triangle, and then picking some edge $uv$ of the current graph, adding a new vertex $w$, and then adding the edges $uw$ and $vw$.

**Question 3.** Is it true that

$$\text{ex}_3(n, G^+) \sim (\sigma(G^+) - 1)\left(\begin{array}{c}n \\ 2 \end{array}\right) \tag{2}$$

for every graph $G$ with tree-width two?

If $G$ is a forest or a cycle, then (2) holds [13, 14] (corresponding results for $r > 3$ were given by Füredi [10]). If $G$ is a graph with $\sigma(G^+) = 2$, then again (2) holds [14]. Proposition 1.1 and Theorem 1.4 give examples of graphs $G$ with $\sigma(G^+) = 4$ and $\text{ex}_3(n, G^+) \geq n^2$.

This leaves the case $\sigma(G^+) = 3$, and in this case, Theorem 1.5 shows that $\text{ex}_3(n, K_{3,t}^+)/n^2 \to \infty$ as $t \to \infty$, even though $\sigma(K_{3,t}^+) = 3$ for all $t \geq 3$. A quadratic upper bound for $\text{ex}_3(n, K_{3,t}^+)$ in Theorem 1.5 is a special case of the following theorem:

**Theorem 1.7.** For every $G$ with $\sigma(G^+) = 3$, $\text{ex}_3(n, G^+) = O(n^2)$.

2 Preliminaries

**Notation and terminology.** A 3-graph is called a **triple system**. The edges will be written as unordered lists, for instance, $xyz$ represents $\{x, y, z\}$. For a set $X$ of vertices of a hypergraph $H$, let $H - X = \{e \in H : e \cap X = \emptyset\}$. If $X = \{x\}$, then we write $H - x$ instead of $H - X$. For
a set $S$ of two vertices in a 3-graph $H$, $N_H(S) = \{x \in V(H) : S \cup \{x\} \in H\}$. The codegree of a pair $S = \{x, y\}$ of vertices in a 3-graph $H$ is $d_H(x, y) = |N_H(S)|$. The shadow of $H$ is the graph $\partial H = \{xy : \exists e \in H, \{x, y, e\} \subset e\}$. The edges of $\partial H$ will be called the sub-edges of $H$. As usual, for a graph $G$ and $v \in V(G)$, $N_G(v)$ is the set of neighbors of $v$ in $G$ and $d_G(v) = |N_G(v)|$.

A 3-graph $H$ is $d$-full if every sub-edge of $H$ has codegree at least $d$.

Thus $H$ is $d$-full is equivalent to the fact that the minimum non-zero codegree in $H$ is at least $d$. The following lemma from [14] extends the well-known fact that any graph $G$ has a subgraph of minimum degree at least $d$ with at least $|G| - (d - 1)|V(G)|$ edges.

**Lemma 2.1.** For $d \geq 1$, every $n$-vertex 3-graph $H$ has a $(d + 1)$-full subgraph $F$ with

$$|F| \geq |H| - d|\partial H|.$$ 

**Proof.** A $d$-sparse sequence is a maximal sequence $e_1, e_2, \ldots, e_m \in \partial H$ such that $d_H(e_1) \leq d$, and for all $i > 1$, $e_i$ is contained in at most $d$ edges of $H$ which contain none of $e_1, e_2, \ldots, e_{i-1}$. The 3-graph $F$ obtained by deleting all edges of $H$ containing at least one of the $e_i$ is $(d + 1)$-full. Since a $d$-sparse sequence has length at most $|\partial H|$, we have $|F| \geq |H| - d|\partial H|$.

## 3 Proofs of Propositions

**Proof of Proposition 1.1.** The proof of the lower bound in Proposition 1.1 is via a random triple system. The idea is to take a random graph not containing a particular graph $G$, and then observe that the triple system of triangles in the random graph does not contain $G^+$. Consider the random graph on $n$ vertices, whose edges are placed independently with probability $p$, to be chosen later. If $X$ is the number of triangles and $Y$ is the number of copies of $G$ in the random graph, then

$$\mathbb{E}(X) = p^3 \binom{n}{3} \quad \mathbb{E}(Y) \leq p^f n^v.$$ 

Therefore choosing $p = 0.1n^{-(v-3)/(f-3)}$, since $f \geq 4$, we find

$$\mathbb{E}(X - Y) \geq 0.0001n^{3-3(v-3)/(f-3)}.$$ 

Now let $H$ be the triple system of vertex-sets of triangles in the graph obtained by removing one edge from each copy of $G$ in the random graph. Then $\mathbb{E}(|H|) \geq \mathbb{E}(X - Y)$, and $G^+ \not\subseteq H$. Select an $H$ so that $|H| \geq 0.0001n^{3-3(v-3)/(f-3)}$. This proves the lower bound in Proposition 1.1 with $a = 0.0001$.

Now suppose $G$ is a bipartite graph with $f$ edges $e_1, e_2, \ldots, e_f$ and $v$ vertices. If a triple system $H$ on $n$ vertices has more than $(n-1)\mathrm{ex}_2(n, G) + (f + v - 1)(\binom{n}{3})$ triples, then by deleting at most $(f + v - 1)(\binom{n}{3})$ triples we arrive at a triple system $H' \subset H$ which is $(f + v)$-full, by Lemma 2.1 and $|H'| > (n - 1)\mathrm{ex}_2(n, G)$. There exists $x \in V(H')$ such that more than $\mathrm{ex}_2(n, G)$ triples of $H'$ contain $x$. So the graph of all pairs $\{w, y\}$ such that $\{w, x, y\} \in H'$ contains $G$. Since every
Proof of Theorem 1.6. Suppose \( e \in V(G) \) such \( e \cap \{z_i\} \in H' \) for all \( i = 1, 2, \ldots, f \), and this forms a copy of \( G^+ \) in \( H' \). \( \square \)

Proof of Proposition 1.2. Let \( G \) be a graph of tree-width two. Then \( G \subset F \), where \( F \) is a graph obtained from a triangle by repeatedly adding a new vertex and joining it to two adjacent vertices of the current graph. It is enough to show \( \text{ex}_3(n,F^+) = O(n^2) \). Suppose \( F \) has \( v \) vertices and \( f \) edges. By definition, \( F \) has a vertex \( x \) of degree two such that the neighbors \( x' \) and \( x'' \) of \( x \) are adjacent. Then \( F' := F - x \) has \( v-1 \) vertices and \( f-2 \) edges. Let \( H \) be an \( n \)-vertex triple system with more than \( (v+f-1)(\frac{n}{2}) \) edges. By Lemma 2.1, \( H \) contains a \((v+f)\)-full subgraph \( H' \). We claim \( H' \) contains \( F^+ \). Inductively, \( H' \) contains a copy \( H'' \) of the expansion of \( F' \). By the definition of \( H' \), \( \{x', x''\} \) has a natural 3-coloring given by \( \phi \) with \( \phi(x'), \phi(x'') \) not in \( F \). Since \( H' \) is full, and therefore if \( z \) is a vertex of the current graph. It is enough to show \( \phi \) is large enough. Let \( n > 1 \) and \( m > \frac{n^2}{3} \). By Lemma 2.1, \( H \) has a \((v+f)\)-full subgraph \( H' \). We claim \( H' \) contains \( F^+ \). Inductively, \( H' \) contains a copy \( H'' \) of the expansion of \( F' \). By the definition of \( H' \), \( \{x', x''\} \) has codegree at least \( v+f \) in \( H' \). Therefore we may select a new vertex \( z \) that is not in \( H'' \) such that \( \{z, x', x''\} \) is an edge of \( H' \), and now \( F \) is embedded in \( H' \) by mapping \( x \) to \( z \). \( \square \)

Proof of Proposition 1.3. Let \( G \) be a 3-colorable planar graph with the given conditions. To show \( \text{ex}_3(n, G^+) = \Omega(n \text{ng}(n,k)) \), form a triple system \( H \) on \( n \) vertices as follows. Let \( F \) be a bipartite \( [\frac{n}{2}] \)-vertex graph of girth \( k \) with at least \( \frac{1}{2}g(\frac{n}{2}, k) \) edges. Let \( U \) and \( V \) be the partite sets of \( F \). Let \( X \) be a set of \( \lfloor \frac{n}{2} \rfloor \) vertices disjoint from \( U \cup V \). Then set \( V(H) = U \cup V \cup X \) and let the edges of \( H \) consist of all triples \( e \cup \{x\} \) such that \( e \in F \) and \( x \in X \). Then

\[
|H| \geq |X| \cdot g(\lfloor \frac{n}{2} \rfloor, k) = \Omega(n \text{ng}(n,k)).
\]

Now \( \partial H \) has a natural 3-coloring given by \( U, V, X \). If \( G^+ \subset H \), then \( G \subset \partial H \) and therefore \( G \) is properly colored, with color classes \( V(G) \cap U, V(G) \cap V \) and \( V(G) \cap X \). By the assumptions on \( G, V(G) \cap (U \cup V) \) induces a subgraph of \( G \) which contains a cycle of length at most \( k \). However, that cycle is then a subgraph of \( F \), by the definition of \( H \), which is a contradiction. Therefore \( G^+ \not\subset H \). \( \square \)

4 Proof of Theorem 1.6

Proof of Theorem 1.6. Suppose \( \text{ex}_2(n, G) = o(n^2) \) and \( |G| = k \), and \( H \) is an \( G^+ \)-free 3-graph with \( |H| \geq (k+1)(\frac{n}{2}) \). By Lemma 2.1, \( H \) has a \( k \)-full-subgraph \( H_1 \) with at least \( n^2/3 \) edges. If \( G \subset \partial H_1 \), then we can expand \( G \) to \( G^+ \subset H_1 \) using that \( H_1 \) is \( k \)-full. Therefore \( |\partial H_1| \leq \text{ex}(n, G) = o(n^2) \). By Lemma 2.1, and since \( |H_1| \geq \delta n^2 \), \( H_1 \) has a non-empty \( n^2-\varphi \)-full subgraph \( H_2 \) if \( n \) is large enough. Let \( H_3 \) be obtained by removing all isolated vertices of \( H_2 \) and let \( m = |V(H_3)| \). Since \( H_3 \) is \( n^2-\varphi \)-full, \( m > n^2-\varphi \). Since \( H_3 \) is \( G^+ \)-free, \( H_3 \subset H_1 \) is also \( G^+ \)-free, and therefore if \( F = \partial H_3, |V(F)| = |V(H_3)| = m \) and \( |F| \leq \text{ex}_2(m, G) = o(m^2) \). So some vertex \( v \) of the graph \( F = \partial H_3 \) has degree \( o(m^{\varphi-1}) \). Now the number of edges of \( F \) between the vertices of \( N_F(v) \) is at least the number of edges of \( H_3 \) containing \( v \). Since \( H_3 \) is
n^{2-\varphi}.full, there are at least $\frac{1}{2}n^{2-\varphi}|N_F(v)|$ such edges. On the other hand, since the subgraph of $F$ induced by $N_F(v)$ does not contain $G$, the number of such edges is $o(|N_F(v)|^\varphi)$. It follows that $n^{2-\varphi} = o(|N_F(v)|^{\varphi-1})$. Since $|N_F(v)| = o(m^{\varphi-1}) = o(n^{\varphi-1})$, we get $2 - \varphi < (\varphi - 1)^2$, contradicting the fact that $\varphi$ is the golden ratio. $\square$

5 Proof of Theorem 1.4

**Proof of Theorem 1.4.** For the upper bound, we repeat the proof of Theorem 1.6 when $F = K_{s,t}$, using the bounds $ex_2(n, K_{s,t}) = O(n^{2-1/s})$ provided by the Kövári-Sós-Turán Theorem [15], except at the stage of the proof where we use the bound on $ex_2(|N_G(v)|, F)$, we may now use

$$ex_2(|N_G(v)|, K_{s-1,t}) = O(|N_G(v)|^{2-1/(s-1)})$$

for if the subgraph of $G$ of edges between $N_G(v)$ contains $K_{s-1,t}$, then by adding $v$ we see $G$ contains $K_{s,t}$. A calculation gives $|H| = O(n^{3-3/s})$.

For the lower bound we must show that $ex_3(n, K_{s,t}^+) = \Omega(n^{3-3/s})$ if $t > (s-1)!$. We will use the **projective norm graphs** defined by Alon, Rónyai and Szabo [2]. Given a finite field $\mathbb{F}_q$ and an integer $s \geq 2$, the norm is the map $N : \mathbb{F}_{q^s-1}^* \rightarrow \mathbb{F}_q^*$ given by $N(x) = x^{1+q+\ldots+q^{s-2}}$. The norm is a (multiplicative) group homomorphism and is the identity map on elements of $\mathbb{F}_q^*$. This implies that for each $x \in \mathbb{F}_q^*$, the number of preimages of $x$ is exactly

$$\frac{q^{s-1}-1}{q-1} = 1 + q + \ldots + q^{s-2}. \quad (3)$$

**Definition 5.1.** Let $q$ be a prime power and $s \geq 2$ be an integer. The projective norm graph $PG(q,s)$ has vertex set $V = \mathbb{F}_{q^s-1}^* \times \mathbb{F}_q^*$ and edge set

$$\{(A,b) : N(AB) = ab\}.$$

**Lemma 5.2.** Fix an integer $s \geq 3$ and a prime power $q$. Let $x \in \mathbb{F}_q^*$, and $A,B \in \mathbb{F}_{q^s-1}^*$ with $A \neq B$. Then the number of $C \in \mathbb{F}_{q^s-1}^*$ with

$$N \left( \begin{array}{c} A+C \\ B+C \end{array} \right) = x \quad (4)$$

is at least $q^{s-2}$.

**Proof.** By (3) there exist distinct $X_1, \ldots, X_{q^{s-2}+1} \in \mathbb{F}_{q^s-1}^*$ such that $N(X_i) = x$ for each $i$. As long as $X_i \neq 1$, define

$$C_i = \frac{BX_i - A}{1 - X_i}.$$ 

Then $(A + C_i)/(B + C_i) = X_i$, and $C_i \neq C_j$ for $i \neq j$ since $A \neq B$. $\square$
Lemma 5.3. Fix an integer \( s \geq 3 \) and a prime power \( q \). The number of triangles in \( PG(q, s) \) is at least \((1 - o(1))q^{3s - 3}/6\) as \( q \to \infty \).

**Proof.** Pick a vertex \((A, a)\) and then one of its neighbors \((B, b)\). The number of ways to do this is at least \(q^{s-1}(q-1)(q^s-1)\). Let \( x = a/b \) and apply Lemma 5.2 to obtain at least \(q^{s-2} - 2\) distinct \( C \notin \{-A, -B\} \) satisfying (4). For each such \( C \), define
\[
c = \frac{N(A + C)}{a} = \frac{N(B + C)}{b}.
\]
Then \((C, c)\) is adjacent to both \((A, a)\) and \((B, b)\). Each triangle is counted six times in this way and the result follows. \(\square\)

For appropriate \( n \) the \( n \)-vertex norm graphs \( PG(q, s) \) (for fixed \( s \) and large \( q \)) have \( \Theta(n^{3-3/s}) \) edges and no \( K_{s,t} \). By Lemma 5.3 the number of triangles in \( PG(q, s) \) is \( \Theta(n^{3-3/s}) \). The hypergraph \( H \) whose edges are the vertex sets of triangles in \( PG(q, s) \) is a 3-graph with \( \Theta(n^{3-3/s}) \) edges and no \( K_{s,t} \). This completes the proof of Theorem 1.4. \(\square\)

6 Proof of Theorems 1.5 and 1.7

We need the following result.

**Theorem 6.1.** Let \( F \) be a 3-uniform hypergraph with \( v \) vertices and \( \text{ex}_3(n,F) < c\binom{n}{2} \). Then \( \text{ex}_3(n, (\partial F)^+) < (c + v + |F|)(\binom{n}{2}) \).

**Proof.** Suppose we have an \( n \) vertex 3-uniform hypergraph \( H \) with \( |H| > (c + v + |H|)\binom{n}{2} \). Apply Lemma 2.1 to obtain a subhypergraph \( H' \subset H \) that is \((v + |F|)-full\) with \( |H'| > c\binom{n}{2} \). By definition, we may find a copy of \( F \subset H' \) and hence a copy of \( \partial F \subset \partial H' \). Because \( H' \) is \((v + |F|)-full\), we may expand this copy of \( \partial F \) to a copy of \( (\partial F)^+ \subset H' \subset H \) as desired. \(\square\)

Define \( H_t \) to be the 3-uniform hypergraph with vertex set \( \{a, b, x_1, y_1, \ldots, x_t, y_t\} \) and 2\( t \) edges \( x_i y_i a \) and \( x_i y_i b \) for all \( i \in [t] \). It is convenient (though not necessary) for us to use the following theorem of the authors [17].

**Theorem 6.2.** ([17]) For each \( t \geq 2 \), we have \( \text{ex}_3(n, H_t) < t^4 \binom{n}{2} \).

**Proof of Theorems 1.5 and 1.7.** First we prove the upper bound in Theorem 1.7. Suppose \( \sigma(G^+) \leq 3 \). This means that \( G \) has an independent set \( I \) and set \( R \) of edges such that \( I \) intersects each edge in \( G - R \), and \( |I| + |R| \leq 3 \). It follows that \( G \) is a subgraph of one of the following graphs (Cases (i) and (ii) correspond to \( |I| = 1 \), Case (iii) corresponds to \( |I| = 2 \), and Case (iv) corresponds to \( |I| = 3 \):
Theorems 6.1 and 6.2 and observe that \( \partial H \)
triples in \( H \)
Similarly, \( H \)
Then the shadow of the set of triples in \( H \)
For the lower bound in Theorem 1.5, we use a slight modification of the construction in Theorem 1.4.
Lemma 2.1, we find a \( N \)
A number of examples of 3-colorable \( G \)
In this paper we studied \( \text{ex}_3(n, G^+) \) where \( G \) is a 3-colorable graph. If \( G \) has treewidth two, then we believe \( \text{ex}_3(n, G^+) \sim (\sigma(G^+))^{n+1} \) (Question 3), and if a planar graph \( G \) has an acyclic 3-coloring, then we believe \( \text{ex}_3(n, G^+) = O(n^2) \) (Question 1). In fact, we also do not know any nonplanar acyclically 3-colorable graph \( G \) with superquadratic \( \text{ex}_3(n, G^+) \). We are also not able to prove or disprove \( \text{ex}_3(n, G^+) = O(n^2) \) when \( G \) is an even wheel (Question 2).
This is equivalent to showing that if \( F \) is an \( n \)-vertex graph with a superquadratic number of triangles, then \( F \) contains every even wheel with a bounded number of vertices.

7 Concluding remarks

- In this paper we studied \( \text{ex}_3(n, G^+) \) where \( G \) is a 3-colorable graph. If \( G \) has treewidth two, then we believe \( \text{ex}_3(n, G^+) \sim (\sigma(G^+))^{n+1} \) (Question 3), and if a planar graph \( G \) has an acyclic 3-coloring, then we believe \( \text{ex}_3(n, G^+) = O(n^2) \) (Question 1). In fact, we also do not know any nonplanar acyclically 3-colorable graph \( G \) with superquadratic \( \text{ex}_3(n, G^+) \). We are also not able to prove or disprove \( \text{ex}_3(n, G^+) = O(n^2) \) when \( G \) is an even wheel (Question 2).

- A number of examples of 3-colorable \( G \) with superquadratic \( \text{ex}_3(n, G^+) \) were given. In particular we determined the order of magnitude of \( \text{ex}_3(n, K_{3,t}^+) \) when near-extremal constructions of
$K_{s,t}$-free bipartite graphs are known. One may ask for the asymptotic behavior of $\text{ex}_3(n, K_{s,t}^+)$ for each $t \geq 3$, since in that case we have shown $\text{ex}_3(n, K_{3,t}^+) = \Theta(n^2)$. Finally, we gave a general upper bound on $\text{ex}_3(n, G^+)$ when $G$ is a bipartite graph, and showed that if $G$ has Turán number much smaller than $n^2$ where $\varphi$ is the golden ratio, then $\text{ex}_3(n, G^+) = O(n^2)$. Determining exactly when $\text{ex}_3(n, G^+)$ is quadratic in $n$ remains an open problem for further research.

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References


