**Problem.** Prove that if two of the angle bisectors of a triangle are equal in length, then the triangle is isosceles.

**Solution.** Let the triangle be $abc$, with angle $2\alpha$ at $a$ and $2\beta$ at $b$, with angle bisectors $ad$ and $be$. Add lines $d\overline{d}$ and $e\overline{e}$ parallel to $ab$. We may assume, without loss of generality, that $\overline{d} \leq \overline{e}$. Notice that $ad\overline{d}$ and $be\overline{e}$ are isosceles triangles with acute angles $\alpha$ and $\beta$ respectively, since $d\overline{d}$ and $e\overline{e}$ are parallel. In these triangles, $\overline{d} \leq \overline{e}$ and $ad = be$, so $\cos \alpha \geq \cos \beta$ and $\alpha \leq \beta$. However $\overline{a}d \leq \overline{b}d$, so (exercise) $ac \leq bc$, which means $\sin 2\alpha \geq \sin 2\beta$ and $\alpha \geq \beta$. Therefore $\alpha = \beta$, and $abc$ is isosceles.

![Figure 1: Add parallel lines $d\overline{d}, e\overline{e}$.](image)

There are many other proofs of this theorem, which is known as the Steiner-Lehmus Theorem, but I have not seen a different one which is shorter, and some of the proofs online really are quite long. As far as I know, this solution is my own.