**Question 1.** A graph is unicyclic if it contains exactly one cycle. Let $G$ be a connected graph with $n$ edges and $n$ vertices. Prove that $G$ is unicyclic. [4]

**Solution 1.** A tree on $n$ vertices is a maximal acyclic graph on $n$ vertices, and we know that a tree on $n$ vertices has $n - 1$ edges. Therefore a graph $G$ with $n$ edges and $n$ vertices definitely contains a cycle. Now we show that $G$ contains only one cycle. If $C_1$ is a cycle in $G$, pick an edge $e_1 \in E(C_1)$ and let $G_1 = G - e_1$. Then $G_1$ is connected and has at least one fewer cycles than $G$. If $G_1$ has a cycle $C_2$, pick an edge $e_2 \in E(C_2)$ and let $G_2 = G_1 - e_2$. Then $G_2$ is connected, with at least one fewer cycles than $G$. Continuing in this way, we eventually reach a graph $G_k = G_{k-1} - e_k$ which has no cycles, but is connected. By definition, $G_k$ is a tree, and $G_k$ has $n$ vertices. Therefore $G_k$ has $n - 1$ edges, which means the number of edges we removed from $G$ was $k = 1$, since $G$ has $n$ edges. Therefore $G - e_1$ is a tree. Now if we add an edge to a tree, we get exactly one cycle, since every two vertices in a tree are joined by a unique path. Therefore $G$ is unicyclic.

**Question 2.** Draw the trees corresponding to the following two Prüfer codes: [6]

$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$
$(0, 1, 0, 7, 7, 4, 4, 5, 1, 5)$

**Solution 2.** Recall, to create a tree from a Prüfer code, start with the code, add the number 0 on the end (the length of the code), and then fill in a sequence of numbers above the code in the following way: above the $i$th element of the code, write the smallest non-zero number which does not appear to the left of it in the numbers written so far in the first row, and does not appear below and to the right in the second row (the original Prüfer code). On the end of the lower sequence we append 0, and then above it right the smallest number not used so far. So given the code $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$, we create two sequences of length 11 and then build a tree on 12 nodes.

$$
\begin{pmatrix}
10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0
\end{pmatrix}.
$$

So the tree is just a path on 12 vertices, labelled $10, 0, 9, 8, 7, 6, 5, 4, 3, 2, 1, 11$ from one end of the path to the other.

The two rows for the second tree are

$$
\begin{pmatrix}
2 & 3 & 6 & 8 & 9 & 7 & 10 & 4 & 11 & 1 & 5 \\
0 & 1 & 0 & 7 & 7 & 4 & 4 & 5 & 1 & 5 & 0
\end{pmatrix}.
$$

Now the tree is easy to draw.
**Question 3.** Find minimum spanning trees in the following weighted graphs, and write down the cost of a minimum spanning tree. In Figure 1(a), all edges on the outer cycle have cost 2, whereas all other edges have cost 1. In Figure 1(b), edges with no costs attached are assumed to have cost 1.

![Figure 1(a) and 1(b)](image)

**Solution 3.** For Figure 1(a), a minimum cost tree consists of all the 7 edges of weight 1 (inner edges) together with any six edges on the outside cycle of weight two which do not create a cycle with the 7 inner edges (there are many ways to choose six such edges). So the minimum cost of a spanning tree is 19.

For Figure 1(b), it’s a bit more complicated, and we’ll do it in the review session.

**Question 4.**

(a) Prove that the graph in Figure 1(b) is not planar.

(b) Let $G$ be the graph obtained from Figure 1(b) by deleting the edge joining the two vertices of degree 10. Prove that $G$ is planar.

(c) Prove that no matter how $G$ is drawn in the plane without crossings, the drawing will have 18 faces.
Solution 4. (a) A corollary to Euler’s Formula is that if a graph on \( n \) vertices is planar, then it has at most \( 3n - 6 \) edges. So if the graph in 1(b) were planar, it would have at most 
\[ 3n - 6 = 3 \times 11 - 6 = 27 \] edges. But we can count how many edges there are either by hand or using the handshaking lemma: from the handshaking lemma,

\[ \# \text{ edges} = \frac{1}{2} \sum_{v \in V} d(v) = \frac{1}{2} (9 \times 4 + 2 \times 10) = 28 \]

since the graph has 9 vertices of degree 4 and 2 of degree 10. But this is more than the 27 edges we’re allowed, so the graph can’t be planar.

(b) We don’t know how to prove a graph is planar, except by actually drawing it in the plane without crossings. Fortunately, in this case it is easy. If we delete the middle edge, we can swing one of the vertices of degree 9 to the outside of the cycle of vertices of degree four to get a drawing with no crossings (see below):

(c) This is Euler’s Formula. The graph \( G \) has 27 edges and 11 vertices, so by Euler’s Formula \( n - e + f = 2 \), we get 
\[ 11 - 27 + f = 2 \] so the number \( f \) of faces in a drawing of \( G \) is always \( f = 18 \).
**Question 5.**

(a) Let \( n \) and \( k \) be positive integers. Use generating functions to determine the number of compositions of \( n \) into \( k \) parts such that all parts are 1 or 2. [3]

(b) Use generating functions to find the number of sequences of integers of length \( k \) which add up to zero such that each entry of the sequence is an element of \( \{-1,1\} \). [3]

**Solution 5.** (a) The generating function for the set \( S \) of sequences \((x_1,x_2,\ldots,x_k)\) where \( x_i \in \{1,2\} \) for \( 1 \leq i \leq k \) is given by

\[
\Phi_S(x) = \Phi_{\{1,2\}}(x)^k = (x + x^2)^k
\]

using the theorem in class and using that the generating function for \( \{1,2\} \) is \( x + x^2 \). To answer the question, we want to find the coefficient \( a_n \) of \( x^n \) in \( \Phi_S(x) \). To do that, we use the binomial theorem:

\[
\Phi_S(x) = (x + x^2)^k = x^k (1 + x)^k = x^k \sum_{j=0}^{k} \binom{k}{j} x^j.
\]

To get \( x^n \), we have to put \( j = n - k \), and then the coefficient of \( x^n \) is

\[
\binom{k}{n-k}.
\]

So there are \( \binom{k}{n-k} \) compositions of \( n \) into \( k \) parts where all the parts are 1 or 2. You can check that this is true for small values of \( n \) and \( k \): for example, when \( n = 7 \) and \( k = 4 \) then the compositions are \((1,2,2,2),(2,2,2,1),(2,1,2,2),(2,2,1,2)\) so there are 4 compositions, which agrees with \( \binom{4}{4} = 4 \).

(b) If we have a sequence \((x_1,x_2,\ldots,x_k)\) which adds up to zero such that \( x_i \in \{-1,1\} \) for all \( i \), then the sequence \((x_1+1,x_2+1,\ldots,x_k+1)\) adds up to \( k \) and \( y_i = x_i + 1 \) is zero or 2. So we have to count compositions \((y_1,y_2,\ldots,y_k)\) of \( k \) such that \( y_i \in \{0,2\} \). The generating function for the set \( S \) of these sequences is

\[
\Phi_S(x) = (1 + x^2)^k
\]

since \( 1 + x^2 \) is the generating function for \( \{0,2\} \). Again we use the binomial theorem:

\[
\Phi_S(x) = \sum_{j=0}^{k} \binom{k}{j} (x^2)^j.
\]

We look for the coefficient of \( x^k \). If \( k \) is odd, then the answer to the problem is zero: there is no \( x^k \) in \( \Phi_S(x) \) in that case because \( 2j \) is always even. Now if \( k \) is even, then put \( j = k/2 \) to get \( \binom{k}{k/2} \) compositions. That is the answer:

\[
\text{Number of compositions} = \begin{cases} 
0 & \text{if } k \text{ is odd} \\
\binom{k}{k/2} & \text{if } k \text{ is even} 
\end{cases}
\]

It makes sense that the answer is zero if \( k \) is odd: we can’t add up an odd number of +1s and -1s and hope to get zero. The answer makes sense for \( k \) even too: we need half of the \( k \) positions to be +1s and the other half to be -1s, and there are \( \binom{k}{k/2} \) ways to choose the \( k/2 \) positions which are going to be +1s.