Note – Working with sums

jacques@ucsd.edu

It is important to know the basic tools for dealing with sums in combinatorics.

A. Multiplying formal power series

Formal power series $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ are multiplied together in the formula

$$A(x)B(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_j b_{k-j} x^k$$

which means the coefficient of x^k in $A(x)B(x)$ is

$$\sum_{j=0}^{k} a_j b_{k-j}.$$

▷ For example, we know that

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{n}{j} \binom{m}{k-j} x^k = \left(\sum_{k=0}^{\infty} \binom{n}{k} x^k \right) \left(\sum_{k=0}^{\infty} \binom{m}{k} x^k \right) = (1+x)^{n+m}$$

by the Binomial Theorem and with $a_k = \binom{n}{k}$ and $b_k = \binom{m}{k}$. Similarly, we have

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} 2^{-j} 3^{j-k} = \left(\sum_{k=0}^{\infty} 2^{-k} \right) \left(\sum_{k=0}^{\infty} 3^{-k} \right) = 3$$

using geometric series and with $a_k = 2^{-j}$ and $b_k = 3^{-j}$.

B. Derivatives and integrals

The derivative of $A(x) = \sum_{k=0}^{\infty} a_k x^k$ is defined by

$$A'(x) = \frac{dA}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

where the new sum starts at $k = 1$. The integral is

$$\int A = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}.$$

When the power series converges absolutely we have for a real number x

$$\int_0^x A(t)dt = \sum_{k=0}^{\infty} \int_0^x t^k dt.$$

▷ For example,

$$\begin{aligned}
\sum_{k=0}^{\infty} k^2 x^k &= x\frac{d}{dx}\sum_{k=1}^{\infty} k x^k \\
&= x\frac{d}{dx}x\sum_{k=1}^{\infty} k x^{k-1} \\
&= x\frac{d}{dx}x\frac{d}{dx}\sum_{k=0}^{\infty} x^k \\
&= x\frac{d}{dx}x\frac{d}{dx}\frac{1}{1-x} \\
&= x\frac{d}{dx}\frac{x}{(1-x)^2} \\
&= \frac{x(1+x)}{(1-x)^3}
\end{aligned}$$

C. Binomial Theorem

For rational a, we have

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k.$$

From this we obtain

$$(t+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k t^{a-k}$$

for any real number t.

▷ For example,

$$\begin{aligned}
(x^2-2x-3)^{-k} &= (x-3)^{-k}(x+1)^{-k} \\
&= (-3)^{-k}(1-x/3)^{-k}(1+x)^{-k} \\
&= (-3)^{-k}\sum_{i=0}^{\infty}\binom{-k}{i}(-x/3)^i \cdot \sum_{j=0}^{\infty}\binom{-k}{j}x^j \\
&= (-3)^{-k}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{-k}{i}\binom{-k}{j}(-3)^{-i}x^{i+j}.
\end{aligned}$$

D. Geometric series

For any positive integer n,

$$\sum_{j=0}^{n} x^j = \frac{x^{n+1} - 1}{x - 1}.$$

Furthermore

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1 - x}.$$

▷ For example,

$$\begin{aligned}
\sum_{n=k}^{\infty} x^{5n+4} &= x^4 \sum_{n=k}^{\infty} (x^5)^n \\
&= x^4 \cdot x^{5k} \sum_{n=0}^{\infty} x^{5n} \\
&= \frac{x^{5k+4}}{1 - x^5}.
\end{aligned}$$

E. Manipulating sums

Interchanging the order of summation is reminiscent of interchanging the order of integration in a multiple integral. For example, if we sum a function $f(i, j)$ over $\{(i, j) : 0 \le i \le j \le n\}$ then we are really saying

$$\sum_{i=0}^{n} \sum_{j=i}^{n} f(i, j).$$

To interchange the order, we get

$$\sum_{j=0}^{n} \sum_{i=0}^{j} f(i, j).$$

We could plot the points (i, j) in the given region in the plane to help change the order of integration.

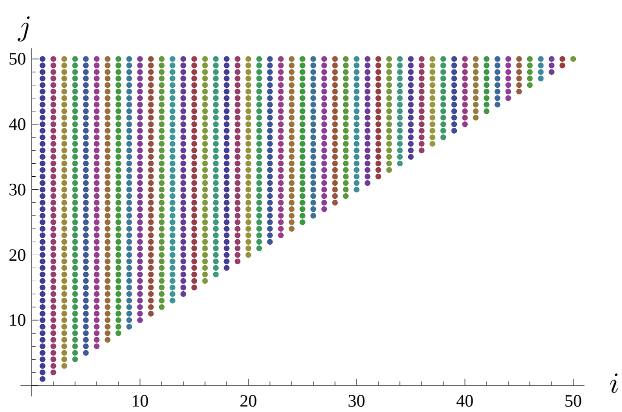

Figure : Range of summation $\{(i, j) : 0 \le i \le j \le n\}$ for $n = 50$.

▷ For example,

$$\begin{aligned}
\sum_{j=0}^{\infty}\sum_{i=j}^{\infty}\binom{i}{j}(-x)^{i-j}x^{2j} &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}x^{2j} \\
&= \sum_{i=0}^{\infty}(-x+x^2)^i \\
&= \frac{1}{1+x-x^2}.
\end{aligned}$$

G. Exercises

Evaluate the following sums in closed form. Throughout, n is a positive integer:

(a) $\sum_{j=1}^{\infty} \frac{x^j}{j}$
(b) $\sum_{j=1}^{\infty} j^2 x^j$
(c) $\sum_{k=0}^{\infty} \sum_{j=0}^{k} (k-j)\binom{n}{j} x^j$
(d) $\sum_{k=0}^{n} k x^{3k+1}$
(e) $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} jk x^{1+2k+3j}$
(f) $\sum_{k=0}^{\infty} \sum_{j=k}^{\infty} x^{j-k}$
(g) $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{k}\binom{k}{j} x^{j+k}$
(h) $\sum_{|j|+|k|\leq n} x^{|j|+|k|}$

Determine a series expansion for each of the following closed form expressions. Throughout n is a positive integer and a is a rational number:

(a) $(n+x)^a$
(b) $\frac{1+x}{1-x}$
(c) $(1+2x+3x^2+2x^3+x^4)^{-1}$
(c) $\frac{x}{x^2-2x-3}$
(d) $\frac{x^3+1}{x^2+1}$
(e) $\frac{1}{x^2-2x-3}$