Unavoidable Cycle Lengths in Graphs

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Abstract

An old conjecture of Erdős states that there exists an absolute constant c and a set S of density zero such that every graph of average degree at least c contains a cycle of length in S. In this paper, we prove this conjecture by showing that every graph of average degree at least ten contains a cycle of length in a prescribed set S satisfying $|S \cap \{1, 2, \ldots, n\}| = O(n^{0.99})$.

1 Introduction.

A set S of integers is called *unavoidable* if there exists an absolute constant c such that every graph of average degree at least c contains a cycle of length in S. If S is not unavoidable, then S is said to be *avoidable.* A simple probabilistic argument (see [2] page 35) shows that for each pair of positive integers d, g, there exists a graph of average degree at least d and girth at least g. This implies that every finite set of integers is avoidable. It is also easily seen that the set $2\mathbb{Z}$ of even integers is unavoidable, whereas the set $2\mathbb{Z}+1$ of odd integers is avoidable.

Bollobás [4] was the first to show that unavoidable sets of arbitrarily small density exist by showing, for $k, m \in \mathbb{Z}$, that $k\mathbb{Z}+m$ is unavoidable if and only if it contains an even integer. This result has been strengthened by a number of authors (see Bondy and Vince [5] and also [16]). Erdős conjectured (see Chung [6], Problem 73) the existence of an unavoidable set of upper density zero. In this paper, we verify this conjecture by proving the following theorem:

Theorem 1. *There exists a set S such that $|S \cap \{1, 2, \ldots, n\}| = O(n^{0.99})$ and any graph of average degree at least ten contains a cycle of length in S.*

Erdős and Gyárfás [8] proposed the existence of a graph of minimum degree at least three containing no cycle of length equal to a power of two, but no example is known.

On the other hand, an *avoidable* set of even numbers of upper density one was brought to the author's attention by Amelie Berger [3]. We briefly outline the construction. First, we write $C(G)$ for the set of lengths of cycles in a graph G, $[n]$ instead of $\{1, 2, \dots, n\}$, and $[m, n]$ instead of $\{m, m+1, \dots, n\}$. Now by a well-known result of Sauer [15], there exists a k-regular graph of girth g on at most k^g vertices. Using this result, we construct graphs $G_2, G_3, G_4, \dots$ such that $|G_k| = n_k$, G_2 is a 4-cycle, G_k is k-regular, G_k has girth larger than $2n_{k-1}$ and $k^{2n_{k-1}} \leq n_k \leq 2k^{2n_{k-1}}$. Now let

$$S = \bigcup_{k=2}^{\infty} [n_k + 1, 2n_k - 1].$$

Since G_{k+1} has girth at least $2n_k$ and G_k has order n_k, $C(G_k) \cap S = \emptyset = C(G_{k+1}) \cap S$ for all $k \geq 2$. Therefore S is avoidable. Furthermore, as $|G_k| \geq 2^{n_{k-1}}$ for all $k \geq 3$, S has upper density one. In contrast, S has lower density zero. We therefore raise the following conjecture:

Conjecture 1.

Every set of positive even integers of positive lower density is unavoidable.

In other words, we conjecture that if S is a set of positive even integers of density $\delta > 0$, then there is a constant $c(\delta)$ depending only on δ such that every graph of average degree at least $c(\delta)$ contains a cycle whose length is an element of S. Evidence for this conjecture is given by the main theorem in [16], where it was shown that every graph of average degree at least $4c$ contains c cycles of consecutive even lengths when c is a positive integer.

This paper is organized as follows: in Section 2, we look at subgraphs H of a graph G which guarantee that $C(G)$ contains rich additive structure. In Section 3, we prove that sufficiently connected graphs G contain such subgraphs. Finally, in Section 4, we prove Theorem 1. Similar methods were used in [17] to show that the number of subsets of $[n]$ which are $C(G)$ for some graph G on n vertices is $o(2^n)$, answering a more recent conjecture of Erdős [7].

NOTATION. The following graph-theoretic notation will be used: if G is a graph, then $V(G)$ denotes its vertex set and $E(G)$ its edge set. For $H \subset G$, $\Gamma_G(H)$ is the set of vertices of $V(G) \backslash V(H)$ incident with a vertex of H, $G[H]$ is the subgraph of G induced by $V(H)$, and $G - H = G[V(G) \backslash V(H)]$. We write $G - E(H)$ for the subgraph of G spanned by the edges of $E(G)\backslash E(H)$. If G_1 and G_2 are subgraphs of a graph G, then $G_1 \cup G_2$ denotes the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If E is a set of pairs in $V(G)$, then $G + E$ denotes the graph on vertex set $V(G)$ and edge set $E(G) \cup E$. Any further notation or terminology will be defined in the context in which it is required.

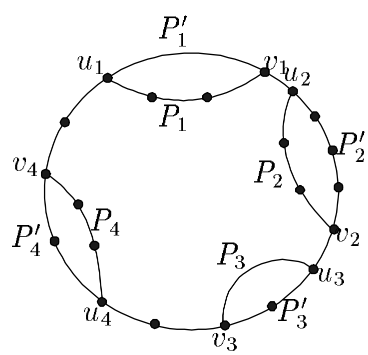
2 Subgraphs with Rich Cycle Structure.

For any multisets A and B, we write $A + B = \{a + b : a \in A, b \in B\}$. If $A = \{a\}$, then we write $a + B$ instead of $A + B$. The set A^* denotes the set of subset sums of A, namely

$$A^* = \{\varepsilon_1 a_1 + \ldots + \varepsilon_n a_n : a_i \in A \text{ and } \varepsilon_i \in \{0, 1\}\}.$$

In this section, we consider graphs G which guarantee that $C(G) \subset [n]$ contains $A + B$ or A^*, where A and B are relatively large sets in $[n]$, or A is a large multiset in $[n]$, respectively.

A *k-truncation* is a plane graph comprising an oriented cycle C, distinct vertices $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$, appearing in this order on C, and, for $i \in [k]$, vertex-disjoint u_i-v_i paths P_i, all internally disjoint from C. A k-truncation is *proper* if, for all $i \in [k]$, the length of P_i is distinct from the length of the subpath P_i' of C from u_i to v_i. In particular, we note that a k-truncation, in which each path P_i is a chord of C, is proper.

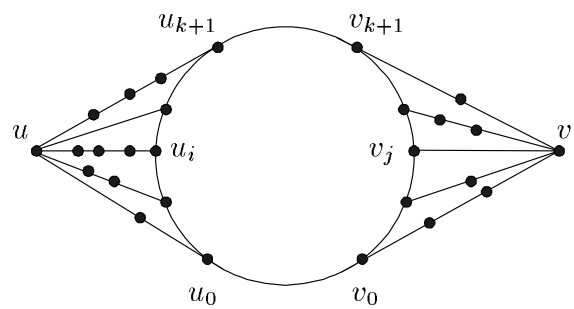


A proper 4-truncation.

Lemma 2.1. *Let G be a proper k-truncation. Then $C(G) \supset a + A^*$, where A is a multiset of size at least k.*

Proof. As G is a proper k-truncation, we may assume $a_i = |E(P_i')| - |E(P_i)| > 0$, otherwise we swap $P_i' \subset C$ with P_i on C. Let $a = |C| - \sum_{i=1}^{k} a_i$ and $I \subset [k]$. Then G contains a cycle of length $a + \sum_{i \in I} a_i$. Let A be the multiset comprising the integers $a_1, a_2, \ldots, a_k$. Then $C(G) \supset a + A^*$. ∎

Let C be an oriented cycle and let $u_0, u_1, \ldots, u_{k+1}, v_{k+1}, v_k, \ldots, v_0$ be distinct vertices of C, appearing in this order. A *k-net* is a graph consisting of C, two vertices u, v not on C, and internally disjoint u-u_i paths P_i^u and v-v_j paths P_j^v, so that P_i^u is disjoint from P_j^v, for $i, j \in [0, k+1]$, and all paths P_i^u and P_j^v are internally disjoint from C.



A 3-*net.*

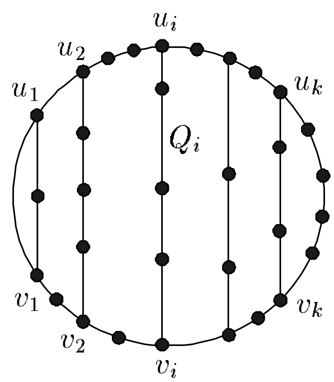
Lemma 2.2. *Let G be a k^2-net. Then $C(G) \supset A + B$, where $|A| \geq |B| \geq k$.*

Proof. Let $m = k^2$ and let P^u be a u_0-u_{m+1} subpath of C, containing all the vertices u_i. Set $H_u = \bigcup_{i=0}^{m+1} P_i^u \cup P^u$, $a^u = |E(P^u)|$, $a_i^u = |E(P_i^u)|$ and let b_i^u be the length of a u_0-u_i subpath of P^u. Then there are u_0-u_{m+1} paths of all lengths in the sets

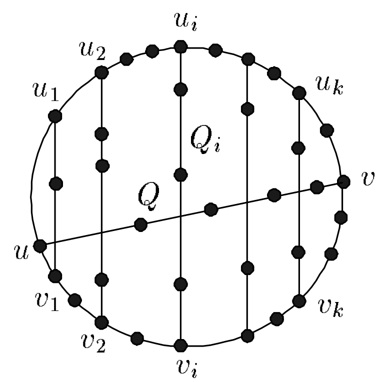
$$A_1^u = \{b_i^u + a_i^u + a_{m+1}^u : i \in [0, m]\} \quad \text{and}$$
$$A_2^u = \{a_0^u + a_i^u + a^u - b_i^u : i \in [m+1]\}.$$

By the Erdős-Szekeres Theorem [9], there is a decreasing or non-decreasing sequence of b_i^u, of length at least k. So at least one of the sets above has size at least k. Similarly, using all the definitions above, with u replaced by v, at least one of A_1^v and A_2^v has size at least k. Without loss of generality, we suppose $|A_1^u| \geq |A_1^v| \geq k$. Let $c = |C| - a^u - a^v$, write $c = a_0 + b_0$, where $a_0 \geq b_0 \geq 1$, and set $a_0 + A_1^u = A$ and $b_0 + A_1^v = B$. For any $a \in A_1^u$ and $b \in A_1^v$, we observe that G contains a cycle of length $a + b + a_0 + b_0 \in A + B$. So $C(G) \supset A + B$. ∎

Let C be a cycle containing vertices $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ in this order along C. A k-*ladder* consists of C together with k pairwise vertex-disjoint paths Q_i between u_i and v_i such that $V(Q_i) \cap V(C) = \{u_i, v_i\}$. A k-*crossladder* is a k-ladder H together with a path Q such that $V(H) \cap V(Q) = \{u, v\}$ for some vertices u and v on the subpaths of C from v_k to u_1 and u_k to v_1, respectively.



A 5-ladder.

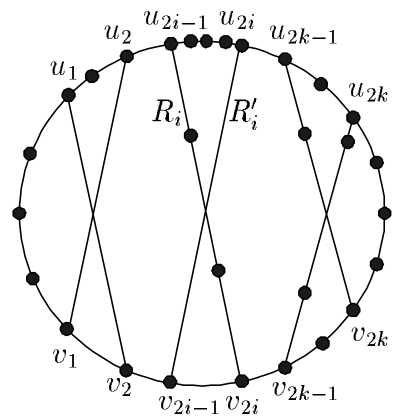


A 5-crossladder.

Lemma 2.3. *Let G be a $4k$-crossladder. Then $C(G) \supset a + A^*$, where A is a multiset with at least k elements.*

Proof. Suppose $V(Q) \cap V(C) = \{u, v\}$, and let Q_{uv} and Q_{vu} be internally disjoint subpaths of C from u to v and v to u respectively. In other words, Q_{uv} contains $u_1, u_2, \ldots, u_k$ and Q_{vu} contains $v_1, v_2, \ldots, v_k$. Let b_i^u be the number of edges in the subpath of Q_{uv} from u_i to u_{i+1}, and let $c_i = |E(Q_i)| + |E(Q_{i+1})|$, for $i \in [4k]$. Define b_i^v similarly, with u_i replaced by v_i. Then, for any $I \subset \{1, 3, \ldots, 4k-1\}$, G contains a cycle of length $|E(Q)| + |E(Q_{uv})| + \sum_{i \in I}(b_i^v + c_i - b_i^u)$, and also a cycle of length $|E(Q)| + |E(Q_{vu})| + \sum_{i \in I}(b_i^u + c_i - b_i^v)$. For each $i \in I$, at least one of the summands in the above expressions is positive. Therefore, by the pigeonhole principle, there exists a set $I' \subset \{1, 3, \ldots, 4k-1\}$ such that $|I'| \geq k$ and (without loss of generality) $a_i = b_i^u + c_i - b_i^v > 0$ for $i \in I$. With A comprising $a_1, a_2, \ldots, a_k$ and $a = |E(Q)| + |E(Q_{vu})|$, we have $C(G) \supset a + A^*$, as required. ∎

Let C be an oriented cycle, and let $u_1, u_2 \ldots, u_{2k}, v_{2k}, v_{2k-1}, \ldots, v_1$ be vertices of C, appearing in this order. A k-*mesh* consists of C, together with u_{2i-1}-v_{2i} and u_{2i}-v_{2i-1} paths R_i and R_i', such that R_i and R_j' are vertex disjoint and internally disjoint from C, for $i, j \in [k]$. Let S_i and S_i' be subpaths of C from u_{2i-1} to u_{2i} and from v_{2i} to v_{2i-1}, respectively. A k-mesh is *proper* if $|E(R_i) + |E(R_i')| \neq |E(S_i)| + |E(S_i')|$.



A proper 3*-mesh.*

Lemma 2.4. *Let G be a proper k-mesh. Then $C(G) \supset a + A^*$, where A is a multiset containing at least k elements.*

Proof. Let $b_i = |E(R_i)| + |E(R_i')|$, $c_i = |E(S_i)| + |E(S_i')|$, and $a = |C| - \sum c_i$. As G is proper, we may assume $b_i - c_i > 0$, otherwise swap R_i, R_i' on C with S_i, S_i', to obtain a new k-mesh. For any set $I \subset [k]$, we have $a + \sum_{i \in I}(b_i - c_i) \in C(G)$. If A comprises $a_1, a_2, \dots, a_k$, then $C(G) \supset a + A^*$. ∎

The following corollary follows easily from the above lemmas, as a set of the form $a + A^*$, where A is a multiset of size $2k$, contains a sum of two sets, each of which has size at least k.

Corollary 2.5. *Let G be a k-crossladder, k^2-net, proper k-mesh or proper k-truncation. Then $C(G) \supset A + B$, where $|A|, |B| \geq \lfloor k/8 \rfloor$.*

For convenience, k-crossladders, k^2-nets, proper k-meshes and proper k-truncations in a graph G will be referred to collectively as k-subgraphs of G.

3 Longest Cycles in Graphs.

In this section, we discuss some structural properties of graphs, relative to their longest cycles, which will lead to the proof of Theorem 1. The idea of this section, essentially, is to show that if G is a graph of average degree at least ten, then G contains a k-subgraph J such that every cycle in J has length $O(k^r)$ for some positive absolute constant r (we will see that $r \leq 24$). We will prove this by finding either a cycle C in G with many chords, in which case we prove $J \subset G[C]$, or a cycle C

such that $G - C$ has many components. In the latter case, we will find paths with internal vertices in distinct components of $G - C$ which together with C form J. The details are found in Lemma 3.9. Throughout this section, C is a longest cycle in a graph G, and C has an implicit orientation. We write v^+ for the successor of a vertex v on C, and v^- for the predecessor of v on C.

Lemma 3.1. *Let $k \in \mathbb{N}$. If at least k vertices of C have degree at least three, then at least $\frac{4}{9}k$ vertices of C have no neighbours in $G - C$, or $G - C$ has at least $\frac{1}{3}k^{1/2}$ components.*

Proof. We may assume $V(G) \backslash V(C) \neq \emptyset$, otherwise $k > \frac{4}{9}k$ vertices of C have no neighbours in $G - C$, as required. Define a directed graph H, which may have loops and multiple edges, as follows: the vertices of H are the components of $G - C$ and, for components A, B of $G - C$, $\vec{AB}$ is an arc of H whenever there exists a vertex $v \in \Gamma_G(A) \cap V(C)$ such that $v^+ \in \Gamma_G(B) \cap V(C)$. In other words, some neighbour of A on C precedes some neighbour of B on C.

We first observe that H has no loops. This follows from the fact that C is a longest cycle, and therefore no component of $G - C$ may have consecutive neighbours on C. Now we prove that each pair of vertices of H is joined by at most two arcs of H. Suppose, for a contradiction, that there exist $A, B \in V(H)$ and three arcs between A and B in H. At least two of the arcs between A and B have the same direction. Let us suppose that two arcs between A and B are directed from A to B. In this case, there exist $u, v \in \Gamma_G(A) \cap V(C)$ such that $u^+, v^+ \in \Gamma_G(B) \cap V(C)$. Let C' be the cycle comprising a u-v path with all internal vertices in A, a u^+-v^+ path with all internal vertices in B, a path from u^+ to v in C and a path from v^+ to u in C. Then $|C'| \geq |C| + 2$, contradicting the fact that C is a longest cycle. Therefore each pair of vertices of H is joined by at most two arcs of H. We now complete the proof. For $v \in V(C)$, define

$$f(v) = |\{A \in V(H) : v \in \Gamma_G(A) \cap V(C)\}|.$$

Then the number of edges in H is precisely $e(H) = \sum_{v \in V(C)} f(v) \cdot f(v^+)$. As no pair of vertices of H is joined by three arcs, we have $e(H) \leq |H|(|H| - 1)$. If $e(H) = \sum_{v \in V(C)} f(v) \cdot f(v^+) \geq \frac{1}{9}k$, then $|H| \geq \frac{1}{3}k^{1/2}$, as required. If $e(H) < \frac{1}{9}k$, then $f(v) \cdot f(v^+) = 0$ for at least $\frac{8}{9}k$ vertices v of degree at least three on C. This implies that $f(v) = 0$ for at least $\frac{8}{9}k$ vertices v, and therefore at least $\frac{4}{9}k$ vertices of C have no neighbours in $G - C$. ∎

Lemma 3.2. *Let G be a 2-connected graph containing at most one vertex of degree two. Then, for any pair of vertices $u, v \in V(G)$, there exist u-v paths of at least two different lengths.*

Proof. We show that G contains a cycle C' and a pair of adjacent vertices z, z^+ in C', each incident with a chord of C'. For if this holds, then for given pair of vertices $u, v \in V(G)$, we find a pair of vertex-disjoint paths R_1, R_2 from u and v to C', using Menger's Theorem and, between their end-vertices r, s on C', there exists a path in $G[C']$ with at least one chord. Let e be a chord of R and let R' denote an r-s path through e, with $E(R') \backslash E(R) = \{e\}$. Then the two u-v paths $R_1 \cup R_2 \cup R$ and $R_1 \cup R_2 \cup R'$ have different lengths.

We now find a cycle C' in G, as above. Let w be a vertex of G of minimum degree. Let P be a longest path from w, chosen so that some neighbour y of the other end-vertex x of P is as close to w on P as possible. Then the x-y path Q on P and xy form a cycle C'. Let us orient this cycle from y to x along P. As x has degree at least three, there is a neighbour $z \neq y$ of x on Q. Let us verify that every neighbour of z^+ is on Q, by the choice of P. If z^+ has a neighbour v in $G - P$, then a w-z^+ path in $G[P]$ together with the edge z^+v forms a longer path than P. If z^+ has a neighbour v in $P - Q$, then $G[P]$ contains a w-v^+ path P' through vz^+ and xz, which has vv^+ as a chord. Now v is closer to w on P' than y is to w on P, contradicting the choice of P. Therefore C' contains a pair of adjacent vertices, namely z and z^+, each incident with a chord of C'. This completes the proof. ∎

Let K be a component of $G - C$. A set $\{u, v, w, x\}$ of four vertices in $\Gamma_G(K) \cap V(C)$ is *good* if, for all distinct $a, b \in \{u, v, w, x\}$, there exists an a-b path P, with all internal vertices in K, of different length to the subpath of C from a to b in the orientation of C.

Lemma 3.3. *Suppose G has connectivity at least four, girth at least five and minimum degree at least six. Let K be a component of $G - C$. Then, for any $u \in \Gamma_G(K) \cap V(C)$, there are $v, w, x \in \Gamma_G(K) \cap V(C)$ so that $\{u, v, w, x\}$ is good.*

Proof. Suppose some vertex z of K has four neighbours in $\Gamma_G(K) \cap V(C)$. If $u \in \Gamma_G(z) \cap V(C)$, then, as G has girth at least five, $\Gamma_G(z) \cap V(C)$ contains a good set $\{u, v, w, x\}$. If $u \notin \Gamma_G(z) \cap V(C)$, then select a u-z path with all internal vertices in K. Then there exists at most one vertex y in $\Gamma_G(z) \cap V(C)$ for which the path from u to y on C has the same length as P. We may now choose $v, w, x \in \Gamma_G(z) \cap V(C) \backslash \{y\}$ so that $\{u, v, w, x\}$ is good.

So we suppose no vertex of K has at least four neighbours on C. This implies that there exists an endblock B of K of minimum degree at least three, as G has minimum degree at least six. By Menger's theorem, there are independent edges uu', vv', ww', xx', such that $u, v, w, x \in \Gamma_G(K) \cap V(C)$, $v', w', x' \in V(B)$ and $u' \in V(K)$. By Lemma 3.2, applied to B, there are paths in B of two distinct lengths between all pairs of vertices in $\{v', w', x'\}$. Let z' be a cutvertex separating

B from the rest of K, if it exists, otherwise set $z' = u'$. Then there exist two paths of distinct lengths between z' and any vertex in $\{v', w', x'\}$, by Lemma 3.2 applied to B. Together with a path from z to u', we obtain paths of two distinct lengths between any pair of vertices in $\{u', v', w', x'\}$, and therefore between any pair in $\{u, v, w, x\}$. We may choose one of these paths, for each pair in $\{u, v, w, x\}$, having length different to the corresponding subpath of C. Thus $\{u, v, w, x\}$ is good. ∎

In the next four lemmas, Lemmas 3.4–3.7, we assume G is a hamiltonian graph, C is drawn as a convex polygon in the plane, and all chords of C are drawn as straight line diagonals inside C. Two chords of C are said to be *crossing* if they cross inside C in this plane drawing. The next lemma is based on Euler's Formula.

Lemma 3.4. *Let G be a planar graph of girth at least g, in which a hamiltonian cycle C bounds the infinite region. Then C has less than $|C|/(g-2)$ chords.*

Proof. Euler's Formula states $|C| - e(G) + f(G) = 2$, where $f(G)$ is the number of faces in G. Also, if the faces of G are bounded by cycles $F_1, F_2, \ldots, F_{f(G)}$, then $2e(G) = \sum_{i=1}^{f(G)} |F_i| \geq |C| + g(f(G)-1)$. Consequently $g|C| - ge(G) + 2e(G) - |C| + g \geq 2g$. Thus $e(G) < |C| + |C|/(g-2)$, and C has less than $|C|/(g-2)$ chords. ∎

Lemma 3.5. *If G has girth at least five, and contains a set Y of at least $4|C|/9$ vertices of degree at least six, then G has a subgraph G' containing C and a set $X \subset Y$ of at least $|C|/9$ vertices of degree at least five in G', in which each chord of C is crossed by some other chord of C.*

Proof. Colour blue those chords of C in G which cross some other chord of C, and colour all remaining edges in G red. We will show that the subgraph G' of G comprising C, together with all its blue chords, is the required subgraph of G. To do so, we must show that a set $X \subset Y$ of at least $|C|/9$ vertices of G are incident with at least three blue chords of C. Suppose, for a contradiction, that this is not the case. This implies that at least $|C|/3$ vertices of Y of degree at least six in G are incident with at least two red chords of C. We also notice that the subgraph of G consisting of C together with all its red chords is a plane graph. However, the total number of red chords incident with these vertices is then at least $\frac{1}{2} \cdot 2|C|/3 = |C|/3$, contradicting Lemma 3.4. ∎

For the next lemma, we use the following convention: if C is an oriented cycle and P is a subpath of C, then P inherits an orientation from C. For convenience, we write $u < v$ if u precedes v on P. We will now be using the definitions and terminology of the last section. Let J denote a k-subgraph of G and suppose that J consists of a cycle C, vertices u_i and v_i on C as in the last section, and paths in $J - E(C)$. For a set S of vertices of C, we say that J is *rooted in* S if, for all i, $u_i \in S$ or $v_i \in S$.

Lemma 3.6. *Let $k \in \mathbb{N}$, and suppose $G - E(C)$ contains an independent set Z of vertices of degree at least five, with*

$$|Z| \geq c(k) = 4k^3(k+2) + 2k(k^2+2)\binom{4k^2(k+2)}{3}.$$

Then G contains a $4k^2$-ladder rooted in Z or a k-subgraph rooted in Z.

Proof. Partition C into k subpaths, each containing at least $|Z|/k$ vertices of Z. If each of these paths has a chord, then G contains a proper k-truncation rooted in Z. We now suppose there is a subpath P in the partition with no chord. Let I be a maximal matching of $Z \cap V(P)$ into $C - P$ in $G - E(C)$, and let $Z_P = Z \cap (V(P) \backslash V(I))$. By the maximality of I, every vertex of Z_P is adjacent to a vertex of $I - P$. Now each vertex of Z_P has at least three neighbours in $I - P$, and

$$|Z_P| = |Z \cap V(P)| - |I| \geq |Z|/k - |I| \geq 4k^2(k+2) + 2(k^2+2)\binom{4k^2(k+2)}{3} - |I|.$$

We now consider two cases. The first case is $|I| < 4k^2(k+2)$; then

$$|Z_P| > 2(k^2+2)\binom{|I|}{3}.$$

This implies the existence of three vertices u, w, v in $I - P$ with at least $2(k+2)^2$ common neighbours in Z_P. Suppose w appears between u and v on $C - P$. Let us show that G contains a k^2-net. Let x, y be distinct common neighbours of u, w, v in $G - E(C)$, chosen as far apart on $C - \{u, v, w\}$ as possible, and let C' be the cycle consisting of the x-y subpath of $C - \{u, v, w\}$ together with wx and wy. We may now select $k^2 + 2$ edges of $G - E(C)$ between u and C', and between v and C' which, together with C', create a k^2-net, as required. Clearly, the k^2-net is rooted in Z_P.

The second case to consider is $m = |I| \geq 4k^2(k+2)$. We will show that G contains a k-crossladder or $4k^2$-ladder. Let us assume $I = \{v_1u_1, v_2u_2, \ldots, v_mu_m\}$, with $v_1 > v_2 > \ldots > v_m$ on P. By the Erdős-Szekeres Theorem [9], we may suppose that in the path $C - P$,

$$\begin{array}{ll} (1) & u_1 > u_2 > \ldots > u_{k+2} \quad \text{or} \\ (2) & u_1 < u_2 < \ldots < u_{4k^2} \end{array}$$

If (1) holds, then G contains a k-crossladder rooted in Z, consisting of the cycle C' containing the edges u_1v_1 and $u_{k+2}v_{k+2}$, and all the vertices $u_1, \ldots, u_{k+2}$ and

$v_1, \ldots, v_{k+2}$, together with the chords $u_i v_i$ of C', for $i \in [2, k+1]$. If (2) holds, then C together with the edges $u_i v_i$ forms a $4k^2$-ladder rooted in Z. ∎

Lemma 3.7. *Let $k \in \mathbb{N}$, and suppose $|C| \geq 9c(k)$ and G has girth at least five. If $G - E(C)$ contains an independent set Y of at least $4|C|/9$ vertices of degree at least six, then G contains a k-subgraph rooted in Y.*

Proof. By Lemma 3.5, G contains a subgraph G' consisting of C and a set $Z \subset Y$ of at least $|C|/9$ vertices of degree at least five in G', in which each chord of C is crossed by some other chord of C. Note that as $|C| \geq 9c(k)$, $|Z| \geq c(k)$. Suppose $Z = \{u_1, u_2, \ldots, u_{c(k)}\}$ and, for a contradiction, suppose G contains none of the required subgraphs rooted in Z. By Lemma 3.6, with $\ell = 4k^2$, this implies that G' contains a ℓ-ladder, J, rooted in Z. Let us assume that

$$J = C + \{u_j v_j, j \in [\ell]\},$$

and that the vertices u_i, v_i appear in the order $u_1, \ldots, u_\ell$, $v_\ell, \ldots, v_1$ on C. Now let

$$E_i = \{u_j v_j : j \in [2ik+1, 2(i+1)k]\},$$

and let Q_i^u denote the subpath of C between u_{2ik+1} and $u_{2(i+1)k}$, containing all the vertices u_j for $j \in [2ik+1, 2(i+1)k]$. Similarly, we define Q_i^v to be the subpath of C between v_{2ik+1} and $v_{2(i+1)k}$, containing all the vertices v_j for $j \in [2ik+1, 2(i+1)k]$. It is convenient to write $e_i = u_{(2i+1)k} v_{(2i+1)k}$.

As each chord of C in G' is crossed by some other chord of C in G', for each $i \in [2k-1]$, the chord e_i is crossed by some other chord f_i of C in G'. We consider two cases:

(1) For some $i \in [2k-1], f_i$ crosses at least k chords of G'

(2) For all $i \in [2k-1], f_i$ crosses fewer than k chords of G'.

In case (1), the subgraph of G' consisting of f_i and all chords of C crossed by f_i is a k-crossladder, rooted in Z, a contradiction. In case (2), as f_i crosses fewer than k chords of C, for each $i \in [2k-1]$, we must have $f_i \in G'[Q_i^u \cup Q_i^v]$. If f_i is a chord of Q_i^u or Q_i^v for at least k values $i \in [2k-1]$, say for $i \in I$, then $C + \{e_i, f_i : i \in I\}$ is a proper k-truncation in G', rooted in Z, a contradiction. So we assume f_i has one endvertex in Q_i^u and one endvertex in Q_i^v for at least k values of $i \in [2k-1]$, say for $i \in I'$. However, $C + \{e_i, f_i : i \in I'\}$ is a k-mesh in G', rooted in Z. Since G' has girth at least five, this k-mesh is proper, as required. ∎

Lemma 3.8. *Let $k \in \mathbb{N}$, and let G be a graph on at least $c(k)$ vertices, of girth at least five. Suppose that $G - E(C)$ contains complete graphs $H_1, H_2, \ldots, H_{c(k)}$ of order* *four, and vertices $u_i \in H_i$, $i \in [c(k)]$ forming an independent set Z in $G - E(C)$. Then G contains a k-subgraph rooted in Z.*

Proof. For a contradiction, suppose G contains no k-subgraph rooted in Z. By Lemma 3.6, this implies that G contains a $4k^2$-ladder H, rooted in the set $Z = \{u_1, u_2, \dots, u_{c(k)}\}$. Let us write $H = C + \{u_i v_i, i \in [4k^2]\}$ for this ladder. It follows that $u_i v_i \in E(H_i)$ for all i. Define E_i, Q_i^u and Q_i^v as in the proof of Lemma 3.7. As in the proof of Lemma 3.7, we must have $H_i \subset Q_i^u \cup Q_i^v$ for at least k values of i, otherwise G contains a k-crossladder rooted in Z. However, as H_i is a complete graph of order four, some edge of H_i is then a chord of Q_i^u or Q_i^v for at least k values of i. This gives a k-truncation in G, rooted in Z. ∎

Lemma 3.9. *Let $k \in \mathbb{N}$, and let G be a graph of connectivity at least four, girth at least five and minimum degree at least six. If C has length at least $2^6 3^4 c(k)^2$, then G contains a k-subgraph.*

Proof. Let H be a spanning subgraph of G, containing $G[C]$, and precisely one edge between v and $G - C$ for each vertex of C with at least one neighbour in $G - C$. If at least $4|C|/9$ vertices of C have no neighbours in $H - C$, then $H[C] = G[C]$ satisfies the conditions of Lemma 3.7, so $H[C]$ contains one of the required subgraphs. Suppose this is not the case. Then, by Lemma 3.1, $H - C$ has $t \geq \frac{1}{3}|C|^{1/2} \geq 24c(k)$ components, $H_1, H_2, \dots, H_t$. Let $u_1, u_2, \dots, u_t$ be their respective neighbours on C in H, and let $U = \{u_1, u_2, \dots, u_t\}$. By Lemma 3.3, for each $u_i \in U$, we may choose vertices $w_i, x_i, y_i \in \Gamma_G(H_i) \cap V(C)$ so that $\{u_i, w_i, x_i, y_i\}$ is good. Define a new graph

$$F = C \cup \{u_i w_i, u_i x_i, u_i y_i, w_i x_i, w_i y_i, x_i y_i : i \in [t]\}.$$

Now $F[U]$ has average degree at most twelve, since F is a union of $|U|$ complete graphs of order four, and therefore has at most $6|U|$ edges. It is known that in a graph of average degree d with n vertices, there is an independent set of size at least $n/2d$. So we may select an independent set $Z = \{u_1, u_2, \dots, u_r\} \subset U$ in $F[U]$ such that

$$|Z| \;\geq\; \frac{1}{24}|U| \;\geq\; c(k).$$

Since $F[U]$ is an induced subgraph of $F - E(C)$, Z is also an independent set in $F - E(C)$. By Lemma 3.8, F contains a subgraph J which is a k-crossladder, k^2-net, k-mesh or k-truncation, rooted in Z. Let us suppose that the edges of $J - E(C)$ are incident with $u_1, u_2, \dots, u_s \in Z$ – since J is rooted in Z, each edge of $J - E(C)$ is indeed incident with one of the vertices u_i. As Z is an independent set in $F - E(C)$, no edge of $J - E(C)$ is incident with two vertices of Z. We conclude that the edge of J in $F - E(C)$ incident with u_i corresponds to a path $P_i \subset G$ with all internal vertices in H_i and endvertices $u_i \in Z$ and $v_i \in \{w_i, x_i, y_i\}$, for $i \in [r]$. It follows that the paths P_i are internally disjoint. Also, as $\{u_i, w_i, x_i, y_i\}$ is good, we may choose P_i to have different length to the appropriate u_i-v_i subpath of C. Now let

$$K = C \cup \bigcup_{i=1}^{s} P_i.$$

Then K is a k-crossladder, k^2-net, proper k-mesh or proper k-truncation in G, as required. ∎

Before summarising the results of this section, we require the following proposition, which is essentially due to Mader [10]:

Proposition 3.10. *Let G be a graph on at least $2k-1$ vertices, of average degree at least $4k-6$. Then there exists a k-connected subgraph of G, of minimum degree at least $2k-2$.*

The following combines the results of this section and Corollary 2.5. Let us first note that for $k \geq 1$,

$$c(k) = 4k^3(k+2) + 2k(k^2+2)\binom{4k^2(k+2)}{3} < 2^{12}k^{12}.$$

Theorem 3.11. *Let G be a graph of girth at least 2^{120} and average degree at least ten. Then there exists an integer $k \geq 2$ and sets A, B of size k such that*

$$C(G) \cap [2^{120}k^{24}] \supset A + B.$$

Proof. By Proposition 3.10, G contains a 4-connected subgraph of G' of minimum degree at least six. Let C be a longest cycle in G'. As G has girth at least $2^{120} > 2^4 3^4 c(8)^2$, there certainly exists an integer $k \geq 1$ such that

$$2^6 3^4 c(8k)^2 \leq |C| \leq 2^6 3^4 c(8k+8)^2.$$

Here we use the inequality on $c(k)$ preceding the theorem. By Lemma 3.9, G' contains an $8k$-crossladder, $(8k)^2$-net, proper $8k$-mesh or proper $8k$-truncation. By Corollary 2.5, $C(G') \supset A + B$ for some $A, B \subset [2^4 3^4 c(8k+8)^2]$ of size at least k. The proof is complete, as $2^6 3^4 c(8k+8)^2 < 2^{120}k^{24}$ for $k \geq 1$. ∎

4 Unavoidable Sets of Cycle Lengths.

We now give a probabilistic construction of an unavoidable set S of positive integers, such that $|S \cap [n]| = O(n^{0.99})$ for all n. The following preliminary lemmas are required:

Lemma 4.1 *Let $k \geq 2^{24}$ be an integer, and let A and B be subsets of $[n]$ of size k. Then there exist sets $A' \subset A$ and $B' \subset B$, of size at most $2k^{1/4}$, such that $|A' + B'| \geq k^{1/2}/4$.*

Proof. Let A', B' be random subsets of A and B, whose elements are selected uniformly and independently from $[n]$, with probability $k^{-3/4}$, respectively. Then the probability that there exist distinct $a, a' \in A'$ and $b, b' \in B$, with $a + b = a' + b'$, is at most $p^4 k \binom{k}{2} < 1/2$. The probability that A' and B' have size between $pk/2$ and $2pk$ is, by standard concentration results for the binomial distribution [2], at most $4\exp(-k^{1/4}/16)$. So there exist sets $A' \subset A$ and $B' \subset B$ as required. ∎

Results of Ruzsa [14] and Komlós, Sulyok, Szemerédi [12] show that, if A is a set of size k, then there exists a set $A' \subset A$ with $|A| = O(k^{1/2})$ and $|A' + A'| = \Omega(k)$. This is clearly best possible, apart from the values of the implied constants.

Lemma 4.2. *Let $k, n \in \mathbb{N}$, and suppose $n \geq k$. Then there exists a set $S(n,k) \subset [n]$, such that $|S(n,k)| = O(k^{-1/4} n \log_2 n)$, and $S(n,k) \cap (A+B) \neq \emptyset$ whenever $A, B \subset [n]$ and $|A| \geq |B| \geq k$.*

Proof. If $k < (\log_2 n)^4$ or $k < 2^{24}$, set $S(n,k) = [n]$, and the proof is complete. We suppose $k \geq (\log_2 n)^4$ and $k \geq 2^{24}$. Let $S(n,k)$ be a random subset of $[n]$, whose elements are chosen independently with probability $p = 16k^{-1/4}(\log_2 n)$ from $[n]$. As $|S(n,k)|$ has a binomial distribution, with probability p and mean pn, $|S(n,k)| \leq 2pn$ with probability $1 - o(1)$, by standard concentration results for the binomial distribution [2].

We now show that the probability that $S(n,k)$ intersects every $A + B$, with $|A| \geq k$ and $|B| \geq k$, is $1 - o(1)$. By Lemma 4.1, the probability that $S(n,k)$ is disjoint from some $A + B$ is at most the probability that $S(n,k)$ is disjoint from some $A' + B'$ with $|A'|, |B'| \leq 2k^{1/4}$ and $|A' + B'| \geq k^{1/2}/4$. This is at most:

$$\binom{n}{2k^{1/4}}^2 \times (1-p)^{-k^{1/2}/4} \leq 2^{4k^{1/4} \log_2 n} \times \exp(-pk^{1/2}/4) = o(1).$$

So, with probability $1 - o(1)$, $|S(n,k)| \leq 32k^{-1/4} n \log_2 n$ and $S(n,k) \cap (A + B)$ whenever $A, B \subset [n]$ have size at least k. ∎

Lemma 4.3. *Let $\eta, c > 0$. Then there exists a set $S' \subset \mathbb{N}$, such that $|S' \cap [n]| = O(n^{1-\eta/4+\varepsilon})$ for all $\varepsilon > 0$, and for all $n \in \mathbb{N}$, $S' \cap A + B \neq \emptyset$ whenever $A, B \subset [n]$ and $|A| \geq |B| \geq cn^{\eta}$.*

Proof. For $j \in \mathbb{N}$, let $I_j = [2^{j-1}, 2^j - 1]$ and $k_j = \lfloor cj^{-1}2^{\eta(j-1)} - j^{-1} \rfloor$. If $k_j = 0$, let $S(n, k_j) = [n]$. Otherwise, let $S(n, k_j)$ denote the sets guaranteed by Lemma 4.2. We claim that

$$S' = \bigcup_{j=1}^{\infty} S(2^{j+1}, k_j) \cap (I_j \cup I_{j+1})$$

intersects $A + B$ whenever $A, B \subset [n]$ and $|A|, |B| \geq cn^{\eta}$. By Lemma 4.2, it is not hard to see that $|S' \cap [n]| = O(n^{1-\eta/4+\varepsilon})$. Suppose $n \in I_j$ and $A + B \subset [n]$, where A and B are sets of size at least $cn^{\eta} \geq jk_j$. We may assume there exist intervals I_{j_1} and I_{j_2}, with $j_2 \leq j_1 \leq j$, such that $A_1 = A \cap I_{j_1}$ and $B_1 = B \cap I_{j_2}$ have size at least k_j. As $I_{j_1} + I_{j_2} \subset I_{j_1} \cup I_{j_1+1}$, it follows that $S(2^{j_1+1}, k_{j_1})$ intersects $A_1 + B_1$. ∎

Proof of Theorem 1. We show that the set $S = S' \cup [2^{120}]$ is unavoidable, where S' is the set in Lemma 4.3, with $\eta = 1/24$ and $c = 1/32$. Certainly, $|S \cap [n]| = O(n^{1-1/100})$, by taking $\varepsilon \leq 1/2400$ in Lemma 4.3. Let G be a graph of average degree at least ten. As we included the interval $[2^{120}]$ in our set we may assume G has girth greater than 2^{120}. By Theorem 3.11, there exists an integer $k \geq 1$ such that, with $n = 2^{120}k^{24}$, $C(G) \cap [n] \supset A + B$ for some sets A, B of size at least $cn^{-\eta}$. By Lemma 4.3, S intersects $A + B$, and therefore intersects $C(G)$. So S is unavoidable. ∎

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