Vertex-Disjoint Cycles of the Same Length

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Abstract

Corradi and Hajnal proved that a graph of order at least $3k$ and minimum degree at least $2k$ contains k vertex-disjoint cycles. Häggkvist subsequently conjectured that a sufficiently large graph of minimum degree at least four contains two vertex-disjoint cycles of the same length. We prove that this conjecture is correct. In doing so, we give a short proof of the known result that if $k > 2$, there is an integer n_k such that any graph of order at least n_k and minimum degree at least $2k$ contains k vertex-disjoint cycles of the same length.

1 Introduction and Notation

For graphs of order n and minimum degree at least $n/2$, Szemerédi's Regularity Lemma has been used to show that most of the vertices of the graph can be covered with disjoint copies of an arbitrary bipartite graph (see Alon and Yuster [1] and Komlós [5]). In the case of cycles, the following remarkable theorem has recently been proved [6] (also see [5]):

Theorem 1.1 *There exists* n_0 *such that if* $n \geq n_0$ *and* G *is a graph of order* $4n$ *and minimum degree at least* $2n$*, then* G *contains* n *vertex-disjoint* 4-*cycles.*

This resolves an older conjecture of Erdös and Faudree [7]. Corresponding problems for dense bipartite graphs have been investigated by Wang [9].

We are interested in such problems for graphs with few edges – graphs of constant average degree. Corradi and Hajnal [3] proved that a graph of minimum degree at least $2k$ and order $n \geq 3k$ contains k disjoint cycles. When the cycles are required to be of the same length, the problem becomes more difficult. Häggkvist (see [8]) conjectured that if G is a sufficiently large graph of minimum degree at least four, then G contains a pair of disjoint cycles of the same length. In this paper, a short proof of this conjecture is given (Theorem 1.2). Thomassen (see [8] page 141) further conjectured that a sufficiently large graph of minimum degree at least $2k$ contains k disjoint cycles of the same length. Thomassen's conjecture was recently proved for $k > 2$ by Egawa [4]: he proved that if G is a graph of order at least $17k + o(k)$ and minimum degree at least $2k$, then G contains k disjoint cycles of the same length. In this paper we give a short proof of Häggkvist's and Thomassen's conjectures:

Theorem 1.2 *Let k be a natural number. Then there exists n_k such that if G is a graph of minimum degree at least $2k$ and order at least n_k, then G contains k disjoint cycles of the same length.*

Asymptotically in k, our proof of Theorem 1.2 gives a weaker result than Egawa's, as we will prove the theorem with n_k of the form k^{7k} when k is large. However, we obtain a proof of Häggkvist's conjecture that demonstrates the difficulty with the case $k = 2$ when compared with the case $k > 2$. We observe that Theorem 1.2 is best possible in the sense that $K_{2k-1,n-2k+1}$ contains no k disjoint cycles, and make the following new conjecture:

Conjecture 1.3 *Let k be a natural number and let G be a graph of order at least $4k$ and minimum degree at least $2k$. Then G contains k disjoint cycles of the same length.*

The truth of conjecture 1.2 would complement the result of Corradi and Hajnal. The graph obtained from a complete bipartite graph $K_{2k-1,2k-1}$

with $2k-1$ vertices in each class by adding a vertex adjacent to all vertices in $K_{2k-1,2k-1}$ shows that the above conjecture cannot be strengthened. It may even be true that a graph of minimum degree at least $4k$ and order at least $8k$ contains $2k$ disjoint even cycles of the same length, in line with Theorem 1.1.

NOTATION. For a graph G, $V(G)$ and $E(G)$ denote the vertex and edge sets respectively and $e(G) = |E(G)|$. If $A, B \subset V(G)$ are disjoint, (A, B) denotes the set of edges with one end in A and the other in B, $e(A, B) = |(A, B)|$, and $G(A, B)$ is the bipartite subgraph of G spanned by (A, B). Implicitly, $uv \in (A, B)$ means $u \in A$ and $v \in B$. The open neighbourhood of $A \subset V(G)$ is $\Gamma(A) = \{v \in V(G)\backslash A : e(v, A) > 0\}$ where if $A = \{v\}$, we write (v, B) for (A, B). We also write $\Gamma(v)$ for $\Gamma(A)$ and $d(v)$ for $e(v, V(G)\backslash A)$. If G_1 and G_2 are subgraphs of G, $G_1 \cup G_2$ denotes the subgraph of G with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$ and $G_1 \cap G_2$ is defined similarly. $G[A]$ is the subgraph of G induced by A, and we write $G[A, B] = G[A] \cup G(A, B)$. When G is a fixed graph and $A \subset V(G)$, we write $e(A)$ instead of $e(G[A])$. The length of a cycle (path) C is $e(C)$, and a k-cycle (k-path) is a cycle (path) of length k. A path P is called an end-path in a non-empty forest F if P is a path in F in which one end vertex has degree 1 in F, the other has degree at least three in F, and all internal vertices of P have degree two in F. We use the special notation $s(G) = \{v \in V(G) : d(v) = 1\}$, $t(G) = \{v \in V(G) : d(v) = 2\}$ and $u(G) = \{v \in V(G) : d(v) = 0\}$. Henceforth, the word *disjoint* is taken to mean *vertex-disjoint*, unless otherwise specified.

2 Lemmas

The proof of Theorem 1.2 requires three simple lemmas.

Lemma 2.1 *Let $G(A, B)$ be a bipartite graph with $|A| > a|B|^b$. Suppose that $d(v) = b$ for all $v \in A$. Then, for some $r \geq 1$, there exist sets $A_1, \ldots, A_r, W \subset A$ and sets $B_1, \ldots, B_r \subset B$ such that for each*

$i = 1, 2, \ldots, r$, $G(A_i, B_i)$ is a complete bipartite graph with $|A_i| \geq a$ and $|B_i| = b$, the B_i are distinct, the A_i are disjoint, $A = A_1 \cup A_2 \cup \ldots \cup A_r \cup W$ and $|W| < a|B|^b$.

Proof. As $|A| > a|B|^b$, an elementary counting argument shows that G contains a complete bipartite graph $G(A', B')$ with $|A'| \geq a$ and $|B'| = b$. Let A_1 be the largest subset of A for which there is $B_1 \subset B$ with $G(A_1, B_1)$ complete bipartite and $|B_1| = b$. If $A_1, A_2, \ldots, A_{i-1}$ are defined, set $V_i = A \backslash \bigcup_{j<i} A_j$ and let A_i be the largest subset of V_i for which there is $B_i \subset B$ with $|B_i| = b$ and $G(A_i, B_i)$ complete bipartite. We continue until we reach a stage r where no such A_r exists in V_r. Setting $W = V_r$, we have $|W| < a|B|^b$. It is easily seen that all the requirements of the Lemma are satisfied by the A_i, B_i and W. □

Lemma 2.2 *Let $G = G[A, B]$ be a forest with $d(v) \geq 2$ for $v \in A$ where $A \neq \emptyset$, with at most l independent edges in $G[A]$, and at most m disjoint 2-paths in $G(A, B)$ with both end-vertices in A. Then $|s(G)| \geq |A| - 2(l+m)+1$.*

Proof. By induction on the number of vertices in G. If $|G| = 3$, then $|s(G)| = 2 \geq |A| - 2(l + m) + 1$ for $l, m \geq 0$. Thus we suppose that $|G| > 1$. Let P be a u-v end path in G. If $G = P$, then $|V(P) \cap A| \leq 2(l + m) + 1$ otherwise G contains $l + 1$ disjoint edges in $G[A]$ or $m + 1$ disjoint 2-paths in $G(A, B)$. In this case, $|s(G)| = 2 \geq |A| - 2(l + m) + 1$ as required. Hence $G \neq P$. Suppose that $d(u) = 1$ and $Q = P \backslash \{v\}$ contains l' disjoint edges in $G[A]$ and m' disjoint 2-paths with both end-vertices in A. By induction, $|s(G - V(Q))| \geq |A| - |V(Q) \cap A| - 2(l - l' + m - m') + 1$ as $G - V(Q)$ has at most $l - l'$ disjoint edges in $G[A]$ and at most $m - m'$ disjoint 2-paths with both end-vertices in A. Now $|V(Q) \cap A| \leq 2(l' + m') + 1$ as above for P, so $|s(G)| \geq |A| - 2(l + m) + 1$ (as $d(u) = 1$), as required. □

Lemma 2.2 is actually best possible, although this fact will not be needed. We require one more lemma. Bollobás and Thomason [1] gave a short proof of the following fact: a multigraph of order n and size at least $n + c$, $c \geq 1$ contains a cycle of length at most $2(\lfloor n/c \rfloor + 1)\lfloor \log_2 2c \rfloor$. The following lemma is an easy consequence of this:

Lemma 2.3 *Let G be a graph of order n and of girth at least $2(\lfloor n/c\rfloor + 1)\lfloor\log_2 2c\rfloor + 1$. Then $2|s(G)| + |t(G)| + |u(G)| > n - 2c$.*

3 Proof of Theorem 1.2

Proof. We prove Theorem 1.2 with

$$n_k = \min\left\{n \in \mathbb{N} : n > 720k \cdot (k-1)^{2k} \cdot 800^{2k} \cdot (\log_2 n)^{4k}\right\}.$$

It suffices to consider graphs in which no vertices of degree greater than $2k$ are adjacent. Assume that G is such a graph, of order $n \geq n_k$, and that G contains no k disjoint cycles of the same length. Let $\mathcal{C} = (C_i)_{i\geq 1}$ be a collection of disjoint cycles in G, each of length less than $40\log_2 n$, chosen such that C_i is a shortest cycle in $G - \bigcup_{j<i} V(C_j)$ for $i \geq 1$ and $|\mathcal{C}|$ is a maximum. We set $V_1 = V(G) - \bigcup_{\mathcal{C}} V(C)$ and $V_2 = V(G) - V_1$. As $\mathcal{C}$ contains no k disjoint cycles of the same length and $n \geq n_k$, we have

$$(1) \quad |V_2| < 800k(\log_2 n)^2 \leq \left(\frac{n}{720k}\right)^{1/2k}.$$

Let U be the set of vertices of V_1 with at least $2k$ neighbours in V_2. Then $|U| \leq 2k|V_2|^{2k} < n/360$, otherwise $K_{2k,2k} \subset G$ by Lemma 2.1 and therefore G contains k disjoint 4-cycles. Now as U contains the isolated vertices of $G[V_1]$, Lemma 2.3, applied with $c = n/18$, gives:

$$(2) \quad 2|s(V_1)| + |U| + |t(V_1)| \geq |V_1| - n/9.$$

Let U' be the set of vertices of $V_1\backslash U$ with exactly $2k-1$ neighbours in V_2. Apply Lemma 2.1 with $A = U'$, $B = V_2$, $a = 2k+1$ and $b = 2k-1$, and let $G(A_i, B_i) \subset G(A, B)$ denote the complete bipartite graphs guaranteed by Lemma 2.1: $|A_i| \geq 2k+1$, $|B_i| = 2k-1$, the A_i are disjoint, the B_i are distinct, and $\left|U'\backslash\bigcup_{i=1}^{r'} A_i\right| \leq (2k+1)|V_2|^{2k-1}$. Note that as $A_i \subset U'$, $\Gamma(A_i) \cap V_2 = B_i$ for $i = 1, 2, \ldots, r'$. Let U'' be the set of vertices of $V_1\backslash(U \cup U')$ with exactly $2k-2$ neighbours in V_2. By Lemma 2.1, applied in the same way as above, we obtain complete bipartite graphs $G(A_i, B_i) \subset G(U'', V_1\backslash(U \cup U'))$

for $i = r'+1, r'+2, \ldots, r$ with $|A_i| \geq 2k+1$, $|B_i| = 2k-2$, the A_i disjoint, the B_i distinct, $\left|U''\backslash \bigcup_{i=r'+1}^{r} A_i\right| \leq (2k+1)|V_2|^{2k-2}$ and $\Gamma(A_i) \cap V_2 = B_i$.

Since U' and U' are disjoint, A_i and A_j are disjoint for $1 \leq i < j \leq r$. Note that as G contains no pair of adjacent vertices of degree more than $2k$, any vertex in A_i has precisely $2k - |B_i|$ neighbours in V_1 for $i = 1, 2, \ldots, r$.

Claim 1. *For $1 \leq i \leq j \leq r$, $|B_i \cap B_j| \geq 2k-3$. Also, $|B_2| = 2k-2$.*

Proof. Suppose that $|B_i \cap B_j| < 2k-3$, for some i, j with $1 \leq i < j \leq r$. Then there exists a 4-cycle in B_i containing no vertex of B_j. However, $G(A_j, B_j)$ contains $(k-1)$ disjoint 4-cycles and so we have k disjoint 4-cycles in G, a contradiction. If $|B_2| = 2k-1 = |B_1|$, then $|B_1 \cup B_2| \geq 2k$ and we find k disjoint 4-cycles in $G(A_1 \cup A_2, B_1 \cup B_2)$.

For the remainder of the proof, we set $X = \bigcup_{i=1}^{r} A_i$ and $Y = \Gamma(X) \cap V_1$, and $Z = V_1 \backslash (X \cup U)$. Since $n \geq n_k$, it follows from (1) that $|Z| \leq (2k+1)|V_2|^{2k-2} + (2k+1)|V_2|^{2k-1} < n/720$. With the above notation, we have $s(V_1) \subset A_1 \cup Z$ and $t(V_1) \subset (X \backslash A_1) \cup Z$ if $|B_1| = 2k-1$, and $s(V_1) \subset Z$, $t(V_1) \subset X \cup Z$ if $|B_1| = 2k-2$. Therefore we deduce the following from (1) and (2):

$$(3) \qquad |X| + |A_1|(|B_1| - 2k + 2) \ > \ 31n/36.$$

Claim 2. *For $1 \leq i \leq r$, $e(A_i) = 0$ and, if $r \geq 2$, no two vertices of A_i have a common neighbour in V_1. Moreover, if $k = 3$, then $e(X) = 0$.*

Proof. Suppose $k \geq 3$ and $e(X) \neq 0$. If $uv \in (A_i, A_j)$, then there are at least $2k-3 \geq k$ triangles of form uvw with $w \in B_i \cap B_j$, by Claim 1. By the minimality of $\mathcal{C}$, there must exist triangles in $\mathcal{C}$ at each vertex $w \in B_i \cap B_j$. No vertices of any B_i are adjacent as these vertices have degree at least $2k+1$ in G. This implies that there must be precisely one triangle in $\mathcal{C}$ at each vertex $w \in B_i \cap B_j$, so there are at least k disjoint triangles in $\mathcal{C}$, a contradiction. Therefore $e(X) = 0$. The same proof shows that $e(A_i) = 0$ for $i = 1, 2, \ldots, r$. Now suppose $u, v \in A_i$ have a common neighbour $w \in V_1$. Then there exists $x \in B_i$ such that $uwvxu$ is a 4-cycle in G, disjoint from

$(k-1)$ 4-cycles in $G(A_1, B_1)$. So no two vertices of A_i have a common neighbour in V_1.

Claim 3. *If* $|B_1| = 2k-1$*, then* $e(A_i, A_j) = 0$ *whenever* $i = j$ *or* $i = 1$*. Moreover, no two vertices of* X *have a common neighbour in* V_1*.*

Proof. Suppose $e(A_i, A_j) \geq 1$ where $i = j$ or $i = 1$. If $B_l \not\subset B_1$ for some $l \in \{1, 2, \ldots, r\}$, then $|B_1 \cup B_l| \geq 2k$ and $G(A_1 \cup A_l, B_1 \cup B_l)$ contains k disjoint 4-cycles, a contradiction. So $B_l \subset B_1$ for $l = 1, 2, \ldots, r$. This implies $|B_i \cap B_j| \geq 2k-2 \geq k$ for $i = j$ or $i = 1$. Now the same argument as in the proof of Claim 2 applies to show that $\mathcal{C}$ contains k disjoint triangles, a contradiction. It follows that $e(A_i, A_j) = 0$ whenever $i = j$ or $i = 1$. For the second part, if $u, v \in X$ have a common neighbour $w \in V_1$, let x be a common neighbour of u and v in V_2. Then we may find $k-1$ disjoint 4-cycles in $G(A_1, B_1)$, disjoint from $uwvxu$. So no two vertices of X have a common neighbour in V_1.

Claim 4. *If* $k \geq 3$ *or* $r = 1$*, then there exists no pair of disjoint 2-paths in* $G[V_1]$ *with all their end-vertices in* X*.*

Proof. Suppose P and Q are a pair of disjoint 2-paths in $G[V_1]$ with all their end-vertices in X. Then there exist $u, v \in V_2$ such that uPu and vQv are disjoint 4-cycles since, by Claim 1, $|B_i \cap B_j| \geq 2k-3 \geq 2$ for each $i, j : 1 \leq i, j \leq r$. Now we easily find $k-2$ disjoint 4-cycles in $G(A_1, B_1)$, all disjoint from uPu and vQv.

Claim 5. *Let* $G[A, B]$ *be a subgraph of* $G[V_1]$ *with* $d(v) \geq 2$ *in* $G[A, B]$ *for* $v \in A$*. Suppose* $G[A]$ *contains no two disjoint edges and* $G(A, B)$ *contains no two disjoint 2-paths with all their end-vertices in* A*. Then* $|B| > |A| - 3$*.*

Proof. If $G[A, B]$ contains a cycle, then it has length at least $40 \log_2 n \geq 10$, by definition of V_1. However, a cycle of length at least 10 in $G[A, B]$ contains two disjoint edges in $G[A]$ or two disjoint 2-paths with end-vertices in A, a contradiction. So $G[A, B]$ is a forest. By Lemma 2.2, $|B| > |A| - 3$.

Claim 6. *$k = 2$ and $|B_1| = 2$.*

Proof. Suppose first that $k \geq 3$. By Claim 2, $e(X) = 0$ and by Claim 4, $G[X, Y)$ satisfies the hypothesis of Claim 5. So $|Y| > |X| - 3$ and $|V_1| > 2|X| - 3$, contradicting (3). This completes the proof for $k \geq 3$. So we assume $k = 2$ and $|B_1| = 3$. Set $Z = U \cup W$ and note that $|Z| \leq n/240$. We now aim to show that $|V_1| > \max\{3(|A_1| - 1 - 3|Z|), 2|X|\}$. This will lead to a contradiction as follows: since $|V_1| < n$, $|X| < n/2$. Therefore, by (3), $|A_1| > 13n/36$ and $|V_1| > 3(|A_1| - 1 - |Z|) > 3(13n/36 - 1 - n/240) > n$.

First note that by Claim 3, every vertex of X has at least one neighbour in Y and no pair of vertices of X has a common neighbour in Y. Therefore $|Y| \geq |X|$ and so $|V_1| \geq 2|X|$. Now let $A = (\Gamma(A_1) \cap Y) \backslash Z$ and $B = \Gamma(A) \cap Y$. By Claims 3, $A \cup B \subset Y \cup Z$. Also note that every vertex of A has at least two neighbours in B, since each has at most one neighbour in V_2 and one neighbour in A_1, by Claim 3. If $G[A]$ contains two disjoint edges $u_iv_i : 1 \leq i \leq 2$, let $w_ix_iy_i$ be disjoint 2-paths in $G(A_1, B_1)$ with $y_i \in \Gamma(u_i) \cap \Gamma(B_1)$ and $w_i \in \Gamma(v_i) \cap \Gamma(B_1)$. Then $\{u_iv_iw_ix_iy_iu_i : i = 1, 2\}$ is a set of two disjoint 5-cycles in G, a contradiction. Similarly, if $G(A, B)$ contains two disjoint 2-paths with all end-vertices in A, we find two disjoint 6-cycles in G. We conclude that $G[A, B)$ satisfies the hypotheses of Claim 5, and so $|B| > |A| - 3 \geq |A_1| - |Z| - 3$. Therefore $|Y| \geq |A| + |B| - |Z| > 2|A_1| - 1 - 3|Z|$. This gives $|V_1| > 3(|A_1| - 1 - |Z|)$, and we have the required contradiction. Therefore $|B_1| = 2 = k$.

Claim 7. *$r \geq 2$ and $\bigcap_{i=1}^{r} B_i \neq \emptyset$.*

Proof. Suppose $r = 1$. By Claim 3, $G[X, Y)$ satisfies the hypotheses of Claim 5 with $A = X$ and $B = Y$. Consequently, $|Y| > |X| - 3$, so $|V_1| > 2|X| - 3$, contradicting (3). Therefore $r \geq 2$. Suppose $\bigcap B_i = \emptyset$. Since the B_i are pairwise intersecting by Claim 1 and $|B_i| = 2$ for $i = 1, 2, \ldots, r$, we must have $r = 3$. No pair of vertices $u, v \in A_i, A_j$ have a common neighbour w in V_1: otherwise $uwvbu$ is a 4-cycle, with $b \in B_i \cap B_j$, disjoint from any 4-cycle in $G(A_l, B_l)$ where $l \notin \{i, j\}$. Therefore $G[X]$ comprises disjoint edges and isolated vertices, and every vertex of X has at least one neighbour in Y. This implies $|Y| \geq |X|$ and so $|V_1| \geq 2|X|$, which contradicts (3). Thus there exists a vertex in $\bigcap_{i=1}^{r} B_i$.

In what remains, set $\bigcap_{i=1}^{r} B_i = \{b\}$ and $B_i = \{b_i, b\}$ for $i = 1, 2, \ldots, r$.

Claim 8. *$e(X) < 2r^2$.*

Proof. Suppose $e(X) \geq 2r^2$. By Claim 2, there exist four disjoint edges $w_1x_1, \ldots, w_4x_4 \in (A_i, A_j)$ for some $i, j : 1 \leq i < j \leq r$. We claim that the following two propositions must hold:

(a) $\Gamma(A_s) \cap \Gamma(A_t) \cap V_1 = \emptyset$ *for* $(s,t) \neq (i,j)$, $s \neq t$ *and*

(b) $e(A_s, A_t) \leq 1$ *for* $(s,t) \neq (i,j)$, $s \neq t$.

We prove (a) and (b) by contradiction. If (a) is false, then for some $(s,t) \neq (i,j)$, there exist vertices $u \in A_s$ and $v \in A_t$ and a vertex $w \in \Gamma(u) \cap \Gamma(v)$. Without loss of generality, we assume that $w_1x_1, w_2x_2 \in (A_i, A_j)$ with $w_1, x_1, w_2, x_2 \notin \{u, v, w\}$. Supposing $t \neq j$ and $a_t \in A_t \backslash \{v\}$, $wuba_tb_tvw$ and $w_1x_1b_jx_2w_2b_iw_1$ are disjoint 6-cycles, a contradiction. If (b) is false then, for some $(s,t) \neq (i,j)$, there exist disjoint edges $y_1z_1, y_2z_2 \in (A_s, A_t)$ disjoint from, say, w_1x_1, w_2x_2. However, supposing $t \neq j$, $y_1z_1b_tz_2y_2by_1$ and $w_1x_1b_jx_2w_2b_iw_1$ are disjoint 6-cycles, a contradiction. It follows that (a) and (b) must be satisfied.

By (a) and (b), all but at most $2\binom{r}{2}$ vertices of $X\backslash(A_i \cup A_j)$ have exactly two neighbours in Y, and none of these vertices have a common neighbour in Y. This implies that $|\Gamma(X\backslash(A_i \cup A_j))| \geq 2|X\backslash(A_i \cup A_j)| - r(r-1)$. By (a) and (b), all but at most $2\binom{r}{2}$ vertices of $A_i \cup A_j$ have exactly one neighbour in Y. By Claim 2, this implies that $|\Gamma(A_i \cup A_j)| \geq \frac{1}{2}|A_i \cup A_j| - r(r-1)$. Therefore $|Y| \geq \frac{1}{2}|A_i \cup A_j| + 2|X\backslash(A_i \cup A_j)| - 2r(r-1) > 2|X| - \frac{3}{2}|A_i \cup A_j| - 2r^2$. This implies $|V_1| > \frac{3}{2}|X| - 2r^2 > \frac{3}{2}|X| - 2|V_2|^2$, contradicting (3).

We now complete the proof of Theorem 1.2. At least $|X| - 4r^2$ vertices of X have two neighbours in Y. Let $\mathcal{D}$ be a maximal collection of edge-disjoint cycles in $G(X, Y)$. Then

$$(4) \qquad e(X,Y) - \sum_{C \in \mathcal{D}} E(C) < |X| + |Y|$$

and therefore $2(|X| - 4r^2) - |\mathcal{D}| \max_{\mathcal{D}} |C| < |X| + |Y|$.

If $\max_{\mathcal{D}} |C| > 2r$, then there are vertices $u, v, w \in A_i \cap V(C)$ for some $i : 1 \leq i \leq r$. If $P \subset C$ is a path of length at most $|C|/3$ between vertices of A_i, then $C - V(P)$ contains a path Q of the same length as P such that P and Q are disjoint. Now bQb is a cycle of length $|Q| + 1$ disjoint from the cycle $b_i P b_i$ of the same length. So $\max_{\mathcal{D}} |C| \leq 2r$, which gives $|X| - 8r^2 - 2r|\mathcal{D}| < |Y|$.

Suppose that $|\mathcal{D}| > r^3$. Then we obtain cycles $C_1, C_2, \ldots, C_d \in \mathcal{D}$, of the same length, with $d > r^2/2$. We may assume that these are pairwise intersecting, otherwise we have the required pair of disjoint cycles. There must exist a set $I \subset Y$ of at most r vertices with the property that every cycle $C_1, \ldots, C_d$ has a vertex in I. This follows because no two cycles have a vertex of X in common and each cycle has length at most $2r$. Each cycle contributes at least two edges to degrees of vertices of I in $G(X,Y)$, implying that there is $z \in I$ such that

$$e(z, X) > r^2/|I| \geq r.$$

As $G(X,Y)$ is bipartite, all neighbours of I lie in X. So the above inequality implies z has two neighbours in some A_i, contradicting Claim 2. Thus $|\mathcal{D}| \leq 2r^3 + 2r^2$. By (4), this implies $|Y| > |X| - 8r^2 - 4r^3 - 4r^4$, so $|V_1| > 2|X| - 8r^2 - 4r^3 - 4r^4$. This contradicts (3) and completes the proof of Theorem 1.2. $\square$

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