

Permutations and Problems

A permutation of $\{1, \dots, n\}$ is a 1-1, onto mapping of the set to itself. Most books initially use a bulky notation to describe a permutation: The numbers $1..n$ are put on one row and the images of these elements under the permutation are put below. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \text{ describes the permutation which sends } 1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 3.$$

This is quickly abandoned in favor of a 1 line "cycle notation" where the same permutation would be denoted $(1\ 2)$. Groups32 uses cycle notation.

It may be of interest for you to know that Groups32 internally stores permutations in the bulky notation and converts them, for input and output, to the cycle notation.¹

To use permutations, you must enter a sub-package of Groups32 called PERMGRPS.² Notice that both the prompt and the list of commands changes:

CREATE	ELEMENTS	HELP	INFO
INSTALL	LEFT->RIGH	MAIN	MULTIPLY
QUIT	RIGHT->LEF	X	

PERM>>

You can find out what each command as usual: type X (or INFO) and the command. This will give you a description.

Notice that the package can be set to multiply permutations from left-to-right (this is the default) or from right-to-left. For Beachy and Blair you need to set it right->left. It will stay that way for the entire session (unless you explicitly change it).

¹ If anyone feels that it would be useful to allow input and output in the 2-line form, please let me know and I will put that feature in.

² The rationale for doing this as a subpackage is to reduce clutter in the main package.

1. *Multiplying Permutations*

IN THESE EXAMPLES WE MULTIPLY RIGHT TO LEFT

```
PERM>> MULTIPLY
Subgroup of Sn -- what is n? Number 3
    Put in a product of cycles.
    End with a blank line
Cycles (1 2 3) (1 2 3) = (1 3 2 )
Cycles (1 2) (1 3) = (1 3 2 )
Cycles (1 3) (1 2) = (1 2 3 )
Cycles
```

This command will keep multiplying cycles until you put in a blank line. The input is a product of cycles. The output is a product of disjoint cycles.

2. *Subgroup of S_n generated by given permutations*

The subgroup generated by a set of permutations is the smallest subgroup of S_n which contains them. This subgroup can be found by the CREATE command³. Each of the generating permutations is entered on a separate line. The input is terminated by a blank line.

```
PERM>> CREATE
Subgroup of Sn -- what is n? Number 5
    Put in generators as product of cycles.
    End with a blank line
Generator (1 2) (3 4)
Generator (1 3)
Generator
Group is of order 8
A  ( )          B  (2 4 )          C  (1 2 ) (3 4 )
D  (1 2 3 4 )   E  (1 3 )          F  (1 3 ) (2 4 )
G  (1 4 3 2 )   H  (1 4 ) (2 3 )
```

The subgroup generated by $\{(1\ 2)(3\ 4), (1\ 3)\}$ is of order 8 and its elements are listed.

³ It is easy to generate groups too big for manipulation by Groups32. You will receive a notice if your group is too big.

If you wish to use the features of the main package to investigate the group you generated, you must INSTALL it to take the place of one of the groups 1-5. Then you must return to the main package by typing MAIN:

```

PERM>> INSTALL
      Install as table k (1..5)  Number 1
PERM>> MAIN

      CENTER          CENTRALIZER  CHART          CONJ-CLS
      COSETS          EVALUATE    EXAMPLES      GENERATE
      GROUP           HELP        INFO          ISOMORPHISM
      LEFT            NORMALIZER  ORDERS        PERMGRPS
      POWERS          QUIT       RESULT        RIGHT
      SEARCH          STOP        SUBGROUPS     TABLE
      X

G1>>

```

Here I have chosen to have this group installed as group1. When I return to MAIN, the list of commands is printed and the prompt changes to show the current group.

```

G1>> ORDERS    for Group Number 1

Group number 1 of Order 8
  1 elements of order 1:  A
  5 elements of order 2:  B C E F H
  2 elements of order 4:  D G
  0 elements of order 8:

G1>> EVALUATE  (use ' for inverse) bd= H
G1>> EVALUATE  (use ' for inverse) db= C
G1>> POWERS    for element D
A D F G

```

Compare with the letters assigned to permutations in this group. You will see that the results just obtained are consistent:

A	()	B	(2 4)	C	(1 2) (3 4)
D	(1 2 3 4)	E	(1 3)	F	(1 3) (2 4)
G	(1 4 3 2)	H	(1 4) (2 3)		

Notice, for example, that $D = (12\ 3\ 4)$ is of order 4 and its powers are $D^1=D$, $D^2=F$, $D^3=G$, $D^4=A$

3. Products

The product of two permutation groups can be represented using the following idea. First let us look at an example: $\mathbb{Z}_2 \times S_3$. We choose generators for each of the component groups but we choose disjoint sets of numbers. Thus if \mathbb{Z}_2 is represented as a group of permutations of $\{1, 2\}$ we would represent S_3 as a group of permutations on $\{3,4,5\}$

```
PERM>> CREATE
Subgroup of Sn -- what is n?  Number 5
      Put in generators as product of cycles.
      End with a blank line
Generator (1 2)
Generator (3 4 )
Generator (3 4 5)
Generator
Group is of order 12
A ( )                B (4 5 )                C (3 4 )
D (3 4 5 )           E (3 5 4 )                F (3 5 )
G (1 2 )             H (1 2 )(4 5 )           I (1 2 )(3 4 )
J (1 2 )(3 4 5 )    K (1 2 )(3 5 4 )           L (1 2 )(3 5 )

PERM>> INSTALL
      Install as table k (1..5)  Number 1
PERM>> MAIN
```

Once this group has been installed as group 1, and we return to MAIN, we can apply any of the operations on the main menu. We can compare with the built-in numbered groups.

Problems

Create the following groups as permutation groups, install them, and try to discover which numbered group corresponds. You can use the ISOMORPHISM command to try to construct an explicit isomorphism.

1. The subgroup of S_4 generated by $\{(1\ 2)(3\ 4), (1\ 3)\}$ (see above)
2. $\mathbb{Z}_2 \times S_3$ (see above)
3. $\mathbb{Z}_2 \times \mathbb{Z}_4$
4. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
5. A_4