Math 103A – Winter,2001
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Generators and Relations

Groups can often be conventiently described in terms of generators and relations. A group G is generated by a set of elements $S=\{x_1,\ldots,x_k\}$ if G is the smallest subgroup which contains the x_i. We write $G = <x_1,..,x_k>$ and call S a set of generators for G. G must contain inverses of the elements in S and also all products that can be formed of elements in S and their inverses. This set of products is the subgroup generated by S.

The Generators usually satisfy relations. Consider groups that have two generators x and y. A relation is given as and equality: $x^3 = e$, $y^2 = e$ are relations. The SEARCH command in Groups32 can be used to find the groups of orders 1-32 which have a given set of generators satisfying given relations:

```
G1>> SEARCH
Enter distinct generators as a string
e.g.  RS means two generators R and S
    Generators:  xy
Do you want these to generate the entire group? (y or n) Y

 Enter the exact order for each generator.
 Press Enter for no order specified
  X is of order    3
  Y is of order    2

 A relation is of the form  LHS = RHS
 Put in LHS RHS  or  LHS  ( if RHS is e )
       <Press ENTER to quit>

Generators:
  XY
Orders:
  X= 3
  Y= 2

RELATIONS:

-- Pressing ESC will abort the search --

   7  group order =  6  X = C   Y = D
   8  group order =  6  X = B   Y = D
  23  group order = 12  X = B   Y = D
  46  group order = 18  X = C   Y = B
  63  group order = 24  X = B   Y = D
  70  group order = 24  X = F   Y = E
```

We have asked for groups having a generator x of order 3 and a generator y of order 2. We have not imposed any additional relations on these generators. We obtain 6 groups. Let's look at the two groups of order 6:

```
G1>> CHART   Order of Groups (1-32 or 0) Number 6
       7    8*
        There are 2 Groups of order 6
        1 abelian and 1 non-abelian
```

We have seen these groups many times before. Group 7 is isomorphic to $\mathbf{Z}_6$ (the abelian group of order 6) and group 8 is isomorphic to S_3 (the non-abelian group of order 6).

$\mathbf{Z}_6$

Since G is abelian, there is an additional relation between x and y namely yx = xy. We see that every element of G can be written as a product $x^a y^b$ where a is 0,1,2 and b is 0,1. We get 6 distinct products this way. The multiplication of two such products is determined by the additional relation: $x^a y^b x^c y^d = x^a x^c y^b y^d = x^{a+c} y^{b+d}$ where a+c is taken mod 3 and b+d is taken mod 2. We can make a Cayley table using this multiplication. This group is cyclic and xy is an element of order 6.

S_3

(or yx In this case there is also an additional relation between x and y: yx = $x^2 y = x^{-1} y$). We see, again, that every element of G can be written as a product $x^a y^b$ where a is 0,1,2 and b is 0,1. We get 6 distinct products this way. The multiplication of two such products is determined by the additional relation. The relation tells us how to move an x to the left past a y. We leave it as an exercise to the reader to show that: $x^a y^b x^c y^d = x^a x^{2^b c} y^b y^d = x^{a+2^b c} y^{b+d}$ where the exponent of x is taken mod 3 and the exponent of y is taken mod 2. Again we can take find the Cayley table for the multiplication.

Why only two groups of order 6?

Cauchy's Theorem assures us that a group of order 6 must have an element, x, of order 3 and an element, y, of order 2. The subgroup H generated by x is of order 3 and so of index 2. Subgroups of index 2 are always normal. Thus $z = yxy^{-1}$ must be an element of H and it must have order 3. The only possibilities are z = x and $z = x^{-1}$. From this we find that either yx=xy or $yx=x^{-1}y$. The analysis above shows that x and y generate an abelian group of order 6 in the first case and a non-abelian group of order 6 in the second case.

How can we get a group of order > 6?

Here is one of the other groups listed which have two generators, one of order 3 and one of order 2. Notice that the generators are given as elements B and D respectively.

```
  23  group order = 12  X = B   Y = D
```

The matter is mysterious only if you assume that everything in the group can be written as a product $x^a y^b$. This would only give 6 elements.

```
G23>> EVALUATE   (use ' for inverse) a= A
G23>> EVALUATE   (use ' for inverse) b= B
G23>> EVALUATE   (use ' for inverse) bb= C
G23>> EVALUATE   (use ' for inverse) ad= D
G23>> EVALUATE   (use ' for inverse) bd= G
G23>> EVALUATE   (use ' for inverse) bbd= J
```

The only elements obtained by putting a product of “B” in front and a product of “D” in back are the 6 elements A,B,C,D,G,J.

Now yx is DB = E which is not one of the 6 elements. We do not have a relation which allows us to “switch an y past an x”. Neither the subgroup generated by B nor the subgroup generated by D is normal. G is not the product of these two subgroups.

```
G23>> SUBGROUPS   of Group Number 23
  ... wait

   * = Normal subgroup
   Generators             Subgroup
  0  { }                 *{ A }
  1  { D }                { A D }
  2  { I }                { A I }
  3  { K }                { A K }
  4  { B }                { A B C }
  5  { F }                { A F G }
  6  { E }                { A E J }
  7  { H }                { A H L }
  8  { D I }             *{ A D I K }
  9  { B D }             *{ A B C D E F G H I J K L }
```

However, DB = E is an element of order 3. So we do have a relation $(yx)^3 = e$. This group is, in fact, the only one that has generators x of order 3, y of order 2 and satisfies $(yx)^3 = e$:

```
G23>> SEARCH
Enter distinct generators as a string
e.g.  RS means two generators R and S
    Generators:  xy
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 Put in LHS RHS  or  LHS  ( if RHS is e )
       <Press ENTER to quit>

LHS RHS >>  yxyxyx
Generators:
  XY
Orders:
  X= 3
  Y= 2

RELATIONS:
  YXYXYX=  e

-- Pressing ESC will abort the search --

  23  group order = 12  X = B   Y = D
```

The elements of this group can be written as products of B and D, but not with all the B's on the left.

```
G23>> EVALUATE   (use ' for inverse) a= A
G23>> EVALUATE   (use ' for inverse) d= D
G23>> EVALUATE   (use ' for inverse) b= B
G23>> EVALUATE   (use ' for inverse) bd= G
G23>> EVALUATE   (use ' for inverse) bb= C
G23>> EVALUATE   (use ' for inverse) bbd= J
G23>> EVALUATE   (use ' for inverse) db= E
G23>> EVALUATE   (use ' for inverse) dbb= F
G23>> EVALUATE   (use ' for inverse) dbd= L
G23>> EVALUATE   (use ' for inverse) bdb= H
G23>> EVALUATE   (use ' for inverse) bbdb= K
G23>> EVALUATE   (use ' for inverse) bdbb= I
```

[It might be interesting to point out that this group is isomorphic to A_4 which can be generated by x = (1 2 3) and y = (1 2)(3 4) and for which yx = (2 4 3) does have order 3.]