MODULI OF HYPERSURFACES IN $\mathbb{P}^3$

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Abstract. The main goal of this paper is to construct a compactification of the moduli space of degree $d \geq 5$ hypersurfaces in $\mathbb{P}^3$, i.e., a parameter space whose interior points correspond to (equivalence classes of) smooth hypersurfaces in $\mathbb{P}^3$ and whose boundary points correspond to degenerations of such hypersurfaces. Motivated by numerous others (see, for example [KSB88], [Ale96], [Hac04]), we consider a hypersurface $D$ in $\mathbb{P}^3$ as a pair $(\mathbb{P}^3, D)$ satisfying certain properties. We find a modular compactification of such pairs and use their properties to classify the pairs on the boundary of the moduli space.

1. Introduction

One focus in algebraic geometry is the study of moduli spaces of higher dimensional varieties. The aim of this paper is to construct a compactification of the moduli space of degree $d$ surfaces in $\mathbb{P}^3$.

In the case of plane curves, using the Hilbert scheme or GIT techniques, one can find a compactification of the space parameterizing smooth degree $d$ curves in $\mathbb{P}^2$, but the boundary does not have a good modular interpretation. For instance, there are points on the boundary that correspond to several different limits of families of plane curves. In [Has99] for the degree 4 case and [Hac04] for general degree, instead of studying curves $C$, the authors worked with pairs $(\mathbb{P}^2, C)$ and certain allowable degenerations. Remembering the embedding of $C$ in $\mathbb{P}^2$ and extracting certain properties yielded a compactification with a modular interpretation.

This paper stems from the natural generalization of Hacking’s work to the given problem: find a good compactification of the moduli space of degree $d$ surfaces using pairs.

For $d = 5$, the moduli space of smooth surfaces has been understood dating back to the 1970s [Hor73]. There is a distinct difference between this moduli space and that of degree $d$ plane curves: it is not irreducible. When fixing the numerical invariants $K_5 = 5$, $p_g = 4$, and $q = 0$ of quintic surfaces, even in the smooth case, one obtains a moduli space with two components. These details will be further explored in Section 6, but we mention it here to indicate the increase in complexity when passing from curves to surfaces.

To find a meaningful compactification of the moduli space of degree $d$ surfaces, we will follow Hacking’s work and study pairs $(X, D)$ that arise as limits of pairs $(\mathbb{P}^3, S)$. To find a natural polarization, we consider surfaces of degree $d \geq 5$ so that $K_5 + S$ is ample. In fact, we will only consider what we call $H$-stable pairs $(X, D)$ which, among other things, have prescribed singularities and a certain relationship $dK_X + 4D \sim 0$. As in Hacking’s work, this class of pairs is particularly well-behaved. Remembering the embedding of $S$ into $\mathbb{P}^3$ allows us to not only have a modular compactification of a space parameterizing $(\mathbb{P}^3, S)$ but also to classify the pairs appearing on the boundary of the moduli space.

A main result is that the class of pairs defined does actually give a compactification of the moduli space of pairs $(\mathbb{P}^3, S)$.

Theorem 1.1. For odd degree $d$, the moduli space of three-dimensional $H$-stable pairs of degree $d$ is a proper Deligne-Mumford stack.
The oddness of degree $d$ in this result is perhaps surprising. While the result is expected, it does not follow from recent results on moduli of pairs. The main issue is that the class of pairs defined below is not obviously a bounded family. If boundedness was immediately known, [AH11] would imply that the moduli space is a Deligne-Mumford stack and properness would follow from a relatively standard argument using the Minimal Model Program.

Hacon, McKernan, and Xu recently proved a strong result about boundedness of families of certain pairs $(X,D)$. However, it requires the coefficients of the divisors appearing in $D$ to belong to a DCC set. Here, in the definition of H-$\epsilon$ stable pair, one requires that $(X,(\frac{1}{4} + \epsilon)D)$ is slc for $\epsilon$ sufficiently small. However, $\epsilon$ is not bounded from below, so results on boundedness like those in [HMX14b] do not apply. If $\epsilon$ was required to belong to a DCC set, [HMX14b, Theorem 1.1] would apply to show the given pairs belong to a bounded family. Unfortunately, in the proof of properness of the stack in Theorem 1.1, one may have to shrink $\epsilon$ to maintain control of the singularities of the pair.

However, a seemingly unrelated goal of this project was to classify the singular pairs appearing on the boundary of the moduli space. In working on this problem, the classification results gave enough control on the singularities of the boundary of the moduli space for odd degree $d$ to apply another result of Hacon, McKernan, and Xu showing this is a bounded family. In other words, regardless of what set $\epsilon$ lives in, the classification results for odd degree $d$ actually imply boundedness.

One should note that this theorem is not false for even degree $d$, just not known. If H-$\epsilon$ stable pairs of even degree can be shown to be bounded, then Theorem 1.1 is true for all H-$\epsilon$ stable pairs.

In light of this discussion, the following theorems serve two purposes: explicit classification of singular threefolds appearing in the moduli space and a means to achieve boundedness without carefully studying the numbers $\epsilon$ that appear in H-$\epsilon$ stable pairs. Classification is of interest even in the absence of the boundedness implication.

The first result is about ambient spaces $X$ with mild singularities. We show that, if $X$ appearing in an H-$\epsilon$ stable pair of odd degree has canonical singularities, then $X$ is actually $\mathbb{P}^3$. If this is the case, it implies that $D \in \left| -\frac{4}{d}K_{\mathbb{P}^3} \right|$ with $(\mathbb{P}^3, \frac{4}{d}D)$ log terminal, so $X$ determines $D$.

**Theorem 1.2.** Given a three-dimensional H-$\epsilon$ stable pair $(X,D)$ of odd degree $d$, if $X$ has canonical singularities, then either

(a) $X \cong \mathbb{P}^3$,
(b) $X$ is isomorphic to the cone over the Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, or
(c) $X \cong \mathbb{P}(1,1,2,4)$

and $D \in |O_X(-\frac{4}{d}K_X)|$ such that $(X,\frac{4}{d}D)$ is log terminal.

There certainly are other examples of threefolds $X$ with canonical singularities and $-K_X$ ample that do not appear in the previous list, brought to the author’s attention by Paul Hacking. However, in the odd degree case, the boundary is very special.

The next result follows from the study of more complicated singularities. In order to guarantee a proper moduli space, we consider semi log canonical (slc) pairs. But, we show in Section 4 that pairs with strictly slc singularities can only appear in the moduli space of pairs with even degree $d$. Therefore, for odd degree, we only need to consider semi log terminal pairs to construct the moduli space.

**Theorem 1.3.** Given a three-dimensional H-$\epsilon$ stable pair of odd degree $d$, $(X,\frac{4}{d}D)$ is semi log terminal. In other words, no strictly semi log canonical degenerations of $\mathbb{P}^3$ appear on the boundary of the moduli space.

This has interesting consequences in the proof of properness. If we have a family of H-$\epsilon$ stable pairs over a punctured curve, to show properness, we complete the family over the curve, resolve the singularities, and eventually take a log canonical model. In general, the log canonical model
of a log terminal pair is always log canonical. However, this result implies that the log canonical model actually has milder singularities and is log terminal.

1.1. A map of this paper. We begin with preliminary notions needed to define H-\(\epsilon\) stable pairs (Section 2).

In Section 3, we define H-\(\epsilon\) stable pairs and use the existence of minimal models to prove that a family of pairs over a punctured curve can be extended in an essentially unique way, justifying the definition.

In Section 4, we prove a number of technical lemmas about extremal contractions in the minimal model program, classify varieties with strictly log canonical singularities, and build up the necessary machinery to prove Theorem 1.3. Using a careful study of extremal contractions in the minimal model program, we generalize [Ish91, Main Theorem] to show that certain strictly log canonical Fano varieties with a finite number of lc singular points have the structure of a cone over an exceptional divisor with discrepancy \(-1\):

**Theorem 1.4.** Let \(X\) be a projective variety with a finite number of strictly log canonical singularities \(\{p_1, \ldots, p_n\}\) and \(-K_X\) ample. If \(a(E, X) \in \{-1, \mathbb{R}_{\geq 0}\}\) for every exceptional divisor \(E\) over \(X\) with \(\text{center}_X(E) \subset \{p_1, \ldots, p_n\}\), then \(X\) is a cone over a numerically Calabi-Yau variety.

We prove a number of related results on the structure of slc Fano varieties and see how this has implications to boundedness of odd degree pairs. We also discuss canonical and log terminal Fano threefolds as a step toward classifying H-\(\epsilon\) stable pairs.

In Section 5, we classify strictly slc pairs appearing as H-\(\epsilon\) stable pairs (in particular, for odd \(d\), there are none) and further analyze the moduli space in question, proving that it is a Deligne-Mumford stack.

Finally, we compare this compactification to existing compactifications of the moduli space of degree \(d\) surfaces in Section 6, focusing mainly on the case \(d = 5\).

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2. Background and previous work

2.1. Singularities. Singular varieties appear naturally in many contexts and are of particular importance in moduli problems. Therefore, we provide a brief introduction to singularities, following [KM98]. We will work with varieties over \(\mathbb{C}\).

A pair \((X, D)\) is a variety \(X\) with a divisor \(D = \sum a_iD_i\) that is a formal linear combination of prime divisors.

**Definition 2.1.** Let \((X, D)\) be a pair where \(X\) is a normal variety and \(D = \sum a_iD_i\) is a \(\mathbb{Q}\)-linear combination of prime divisors such that \(K_X + D\) is \(\mathbb{Q}\)-Cartier. Let \(f : Y \to X\) be a birational morphism from a smooth variety \(Y\) with exceptional divisors \(E_i\) and strict transform \(f_*^{-1}D = \sum a_i f_*^{-1}D_i\). Then, we can write

\[
m(K_Y + f_*^{-1}D) \sim f^*(m(K_X + D)) + \sum ma_i(E_i, X, D)E_i
\]

or

\[
K_Y + f_*^{-1}D \sim f^*(K_X + D) + \sum a_i(E_i, X, D)E_i
\]

with \(a_i(E_i, X, D) \in \mathbb{Q}\).
The rational number \( a_i(E_i, X, D) \) is called the discrepancy of \( E_i \) with respect to the pair \((X, D)\). If \( F \) is any divisor on \( X \), we define \( a(F, X, D) = -\text{coeff}_FD \), so \( a(D_i, X, D) = -a_i \) and \( a(F, X, D) = 0 \) for \( F \neq D_i \).

Using the discrepancy, one can define the type of singularities of a pair \((X, D)\).

**Definition 2.2.** Let \((X, D)\) be a pair where \( X \) is a normal variety and \( D \) is a sum of distinct prime divisors \( D = \sum a_iD_i, a_i \leq 1 \). Assume that \( K_X + D \) is \( \mathbb{Q} \)-Cartier. Then, \((X, D)\) is

- **terminal**
- **canonical**
- **klt**
- **plt**
- **lc**

\[
\begin{align*}
\text{terminal} & \quad > 0 \text{ for all exceptional } E \\
\text{canonical} & \quad \geq 0 \text{ for all exceptional } E \\
\text{klt} & \quad \text{if } a(E, X, D) > -1 \text{ for all } E \\
\text{plt} & \quad \text{if } a(E, X, D) > -1 \text{ for all exceptional } E \\
\text{lc} & \quad \text{if } a(E, X, D) \geq -1 \text{ for all exceptional } E
\end{align*}
\]

There is one other class of singularities that we will be concerned with: divisorial log terminal or dlt singularities.

**Definition 2.3.** A pair \((X, D)\) where \( X \) is normal, \( D = \sum a_iD_i \) where \( 0 \leq a_i \leq 1 \), and \( K_X + D \) \( \mathbb{Q} \)-Cartier is dlt if there exists a log resolution \( f : Y \to X \) such that \( a(E, X, D) > -1 \) for every exceptional divisor \( E \subset Y \).

Note that this definition only requires one log resolution to give the condition \( a(E, X, D) > -1 \), not all of them. For example, the pair \((\mathbb{P}^2, D)\) where \( D = L_1 + L_2 \), the sum of two lines that intersect transversally, is dlt because the identity map is a log resolution. However, it is not true that \( a(E, X, D) > -1 \) for every exceptional divisor; if \( Y \) is the blow up of \( \mathbb{P}^2 \) at \( L_1 \cap L_2 \) with exceptional divisor \( E \), then \( a(E, X, D) = -1 \).

We will also consider non-normal varieties.

**Definition 2.4.** A variety \( X \) is semi-normal [Kol13, Definition 5.1] if \( X \) is \( S_2 \) and its codimension 1 points are either regular points or double normal crossing points (nodes).

**Definition 2.5.** A pair \((X, D)\) is semi log canonical, slc, (respectively semi log terminal, slt), if

- \( X \) is semi-normal.
- \( K_X + D \) is \( \mathbb{Q} \)-Cartier.
- If \( \nu : X^\nu \to X \) is the normalization of \( X \), \( \Delta^\nu \) the conductor, and \( D^\nu \) the preimage of \( D \), then \((X^\nu, \Delta^\nu + D^\nu)\) is log canonical (respectively, log terminal). (Note this makes sense because \( K_X + \Delta + D^\nu \sim \nu^*(K_X + D) \) [Kol13, 5.7.5].)

### 2.2. Previous work.

We will construct a KSBA compactification of the moduli space of degree \( d \) surfaces in \( \mathbb{P}^3 \) and mention here the relationship between this compactification and related ones.

The KSBA compactification is named for Kollár, Shepherd-Barron, and Alexeev, and is a parameter space for stable objects with semi-log canonical singularities. Our precise definition of stable objects will appear in Section 3. One can consider KSBA compactifications as higher dimensional analogues of \( \overline{M}_{g,n} \), the moduli space of stable genus \( g \) curves with \( n \) marked points.

The framework motivating this definition of the compactification comes from earlier work of Hassett and Hacking, studying moduli spaces of degree \( d \) plane curves by considering them as pairs \((\mathbb{P}^2, C)\). A compactification of the moduli space of the space of pairs \((\mathbb{P}^2, C_4)\), where \( C_4 \) has degree 4 was studied in [Has99] and, more generally, for any degree \( d \geq 4 \), a compactification of pairs \((\mathbb{P}^2, C_d)\) was constructed in [Hac04].

The work in [Hac04] provides much of the foundation for this paper as the study of pairs \((\mathbb{P}^3, S_d)\) is a natural generalization. In a different direction, one could generalize [Hac04] to study pairs \((S, D)\) for other del Pezzo surfaces. In particular, a compactification of the space of pairs \((\mathbb{P}^1 \times \mathbb{P}^1, C_{3,3})\) using similar machinery is described in [DH18].
There is another approach to the study of plane curves and surfaces in \( \mathbb{P}^3 \) using the tools provided by GIT. For quintic surfaces, a boundary divisor in the GIT compactification is described in part in [Ran17].

However, as discussed in [WX14], GIT begins to fail for higher degree \( d \). In the GIT construction, the moduli space depends on the power \( r \) of \( \omega_S \) (or other ample line bundle) being used to embed smooth surfaces into projective space and it is shown that the moduli spaces in this construction do not stabilize. In particular, in [WX14, Theorem 1 (2)], there are families of degree \( d > 30 \) smooth surfaces over a punctured base whose limit does not stabilize as \( r \) increases.

Therefore, it benefits us to approach the problem for general degree \( d \) surfaces using the KSBA framework instead of GIT. Some comparison to the GIT case for degree \( d = 5 \) will be given in Section 6.

3. H-\( \epsilon \) Stable Pairs

3.1. Definition and motivation. We are interested in studying the moduli space of hypersurfaces \( S \) (of a fixed degree) in \( \mathbb{P}^3 \). As motivated in the introduction, instead of studying moduli of such \( S \) directly, we study moduli of pairs \( (X, D) \) where \( X \) is a degeneration of \( \mathbb{P}^3 \) and \( D \) is a degeneration of \( S \).

This was studied in different ways by many authors, first in [Has99] for degree 4 curves and then in [Hac04] for all degree. A compactification was constructed in [Hac04] by considering moduli of pairs \( (X, D) \) where \( X \) was a slc surface that smoothed to \( \mathbb{P}^2 \) and \( D \) was a divisor such that \( dK_X + 3D \cong 0 \) and \( K_X + \left( \frac{d}{3} + \epsilon \right)D \) was ample for some (and hence all) \( \epsilon \) sufficiently small. He was also able to explicitly determine the surfaces \( X \) (and thus the divisors \( D \)) appearing on the boundary of the moduli space.

This was a variant on another construction of compact moduli of such pairs (see [KSB88] and [Ale96]), where the moduli space with some fixed \( \epsilon \in \mathbb{Q} \) was considered.

Now, consider the direct generalization of [Hac04]: a compactification of the moduli space of degree \( d \) hypersurfaces in \( \mathbb{P}^3 \). Unfortunately, much of the work in [Hac04] relies on the existing classification of surface singularities, but the approach of recasting the problem in terms of pairs \( (X, D) \) has its advantages. To this extent (motivated by [Hac04]), we present the following definitions.

**Definition 3.1.** A pair \( (X, D) \), where \( X \) is a threefold and \( D \) is an effective \( \mathbb{Q} \)-Cartier divisor, is said to be \( \epsilon \)-semistable in the sense of Hacking, or H-\( \epsilon \) semistable, of degree \( d \ (d \in \mathbb{N}, d \geq 4) \) if

- \( X \) is normal and log terminal.
- The pair \( (X, \frac{1}{2}D) \) is log canonical.
- \( dK_X + 4D \) is linearly equivalent to zero.
- There is a deformation \( (\mathcal{X}, \mathcal{D})/T \) of \( (X, D) \) over the germ of a curve such that the general fiber \( \mathcal{X}_t \cong \mathbb{P}^3 \) and the divisors \( K_{\mathcal{X}/T} \) and \( D \) are \( \mathbb{Q} \)-Cartier.

**Remark 3.2.** The last condition restricts us to the component of the moduli space parameterizing pairs that admit smoothings to \( \mathbb{P}^3 \). This is not necessary; one could replace this condition with

- \( K_X^3 = K_{\mathbb{P}^3}^3 = -64 \), \( \chi(X, O_X) = \chi(\mathbb{P}^3, O_{\mathbb{P}^3}) \), and \( X \) is Cohen-Macaulay.

Note that the relationship \( dK_X + 4D \sim 0 \) implies that, if \( K_X^3 \) is fixed, so is the volume of the pair \( (K_X + \frac{d}{3}D)^3 \). The condition that \( X \) is Cohen-Macaulay potentially restricts us to some components of the general moduli space, but by [KK10, Corollary 7.13], the components parameterizing non-Cohen Macaulay pairs are disconnected from these components.

Furthermore, studying these pairs will necessarily give a moduli space with more components.

Even for smooth surfaces of degree 5, the moduli space has at least two irreducible components. See Section 6 for further details on this relationship.
Studying $H$-$\epsilon$ semistable has some advantages; in particular, the threefold $X$ is required to be normal, facilitating a simpler study of the divisor $D$. However, they have one distinct disadvantage: the moduli space of $H$-$\epsilon$ semistable pairs is not separated. There are examples of families of log smooth pairs with more than one semistable limit. Therefore, we will primarily concern ourselves with $H$-$\epsilon$ stable pairs, defined below. The moduli space of $H$-$\epsilon$ stable pairs is separated; limits are unique in an appropriate sense (Theorem 3.8).

**Definition 3.3.** A pair $(X, D)$, where $X$ is a threefold and $D$ is an effective $\mathbb{Q}$-Cartier divisor, is said to be $\epsilon$-stable in the sense of Hacking, or $H$-$\epsilon$ stable, of degree $d$ if

- The pair $(X, (\frac{d}{\epsilon} + \epsilon)D)$ is semi log canonical and the divisor $K_X + (\frac{d}{\epsilon} + \epsilon)D$ is ample for some $\epsilon > 0$.
- The divisor $dK_X + 4D$ is linearly equivalent to zero.
- There is a deformation $(\mathcal{X}, \mathcal{D})/T$ of $(X, D)$ over the germ of a curve such that the general fiber $\mathcal{X}_t \cong \mathbb{P}^3$ and the divisors $K_{\mathcal{X}/T}$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.

**Remark 3.4.** In [DH18], for del Pezzo surfaces $S$ and curves $C$, they make the natural generalization of Hacking’s definition and refer to pairs $(S, C)$ as almost $K3$ stable. One could call this definition for threefolds almost CY stable as a generalization of that work.

Considering $H$-$\epsilon$ stable pairs has advantages over semistable pairs, some of which are detailed in the following trivial lemma. Also, it gives a separatedness condition on the moduli space (Theorem 3.9 below).

**Lemma 3.5.** If $(X, D)$ is an $H$-$\epsilon$ stable pair, the following hold:

(a) $K_X$ is anti-ample.
(b) $D$ is ample.
(c) Both $K_X$ and $D$ are $\mathbb{Q}$-Cartier.
(d) If $X$ is strictly slc, the strictly slc locus of $X$ is not contained in the support of $D$.

### 3.2. Limits of $H$-$\epsilon$ stable pairs exist.
First, we prove that we can extend families of $H$-$\epsilon$ semistable pairs (in a not necessarily unique way). To do this, we first have a lemma and technical definition.

**Lemma 3.6.** Let $\mathcal{X}/T$ be a flat family of projective varieties over the germ of a curve such that the general fiber is normal. Let $\mathcal{X}^\times/T^\times$ be the restriction of the family to the punctured curve $T^\times = T \setminus \{0\}$. If $\mathcal{B}$ is a relatively nef $\mathbb{Q}$-Cartier divisor $\mathcal{X}$ such that $\mathcal{B}|_{\mathcal{X}^\times} \sim 0$, then $\mathcal{B} \sim 0$ in $\text{Cl}(\mathcal{X}/T)$.

**Proof.** Let $X_1, X_2, \ldots, X_n$ be the irreducible components of $X = X_0$, the fiber over the closed point. Then, there is an exact sequence

$$0 \to ZX \to \oplus ZX_i \to \text{Cl}(\mathcal{X}/T) \to \text{Cl}(\mathcal{X}^\times/T^\times) \to 0.$$  

Since $\mathcal{B}|_{\mathcal{X}^\times} \sim 0$, we can write $\mathcal{B} \sim \sum a_i X_i$ for $a_i \in \mathbb{Z}$, arranged so that $a_1 \leq a_2 \leq \cdots \leq a_n$. Because $X \sim 0$ in $\text{Cl}(\mathcal{X}/T)$, we can assume $a_i \leq 0$ for all $i$ and $a_1 = 0$. Assume to the contrary that there exists an $i$ such that $0 = a_1 = a_2 = \cdots = a_{i-1} > a_i \geq \cdots \geq a_n$. For each $j \leq i - 1$, and any curve $C \subset X_j$ with $C \not\subset X_k$ for $k \neq j$, we have $X_k \cdot C \geq 0$. Therefore,

$$\mathcal{B} \cdot C = a_i X_i \cdot C + \cdots + a_n X_n \cdot C \leq 0.$$  

But, since $\mathcal{B}$ is relatively nef, this implies $X_i \cdot C = 0$ for $i \leq l \leq n$. However, if there exists an $l$ such that $X_l \cap X_j \neq \emptyset$, then, choosing any curve $C \subset X_j$ that intersects (but is not contained in) $X_l$, we must have $X_l \cdot C > 0$, as it counts the number of points in the intersection. Since $X$ is connected, we must have $X_l \cap X_j \neq \emptyset$ for some $l, j$ such that $i \leq l \leq n$ and $j \leq i - 1$. Therefore, we have a contradiction, so $0 = a_1 = a_2 = \cdots = a_n$ and $\mathcal{B} \sim 0$.  

\[\square\]
Definition 3.7. Let \((X, D)/T\) be a pair consisting of a normal variety \(X\) and an effective Weil divisor \(D\), flat over the DVR \(T\). Let \(X\) be the fiber over the closed point. A semistable log resolution of \((X, D)\) is a proper birational morphism \(f : Y \to X\) such that \(Y\) is smooth, \(\text{Ex}(f)\) is a divisor, \(g_*^{-1}X\) is reduced, and \(\text{Ex}(f) \cup g^{-1}\text{Supp} \mathcal{D} \cup g_*^{-1}(X)\) is a simple normal crossing divisor.

Semistable log resolutions exist (possibly after finite surjective base change) by [KM98, Theorem 7.17].

Using existence of resolutions and the previous lemma, we will first show that families of log smooth pairs \((\mathbb{P}^3, D)\) over a punctured curve can be completed to a family of semistable pairs, Definition 3.1, not necessarily in a unique way. See Remark 3.10 for a summary of this process.

Theorem 3.8. Let \(0 \in T\) be the germ of a curve and write \(T^\times = T - 0\). Let \(D^\times \subset \mathbb{P}^3 \times T^\times\) be a family of smooth hypersurfaces over \(T^\times\) of degree \(d \geq 4\). Then, there exists a finite surjective base change \(S \to T\) and a family \((X, D)/S\) of \(H\)-\(\epsilon\) semistable pairs extending the pullback of the family \((\mathbb{P}^3 \times T^\times, D^\times)/T^\times\) such that the divisors \(K_X\) and \(D\) are \(\mathbb{Q}\)-Cartier.

Proof. Complete \((\mathbb{P}^3 \times T^\times, D^\times)\) to a flat family \((\mathbb{P}^3 \times T, D)\) over \(T\). Possibly after base change, which we suppress in the notation, there is a semistable log resolution \(\pi : (X_1, D_1) \to (\mathbb{P}^3 \times T, D)\) which is an isomorphism over \(T^\times\).

Now, run a \(K_{X_1} + \frac{4}{d}D_1\) MMP over \(T\). Let \((X_2, D_2)/T\) denote the end product. Since it is the end product of an MMP, \((X_2, X + \frac{4}{d}D_2)\) is dlt and \(\mathbb{Q}\)-factorial. Also, \(K_{X_2} + \frac{4}{d}D_2\) is relatively nef. Because \((\mathcal{X}_1, D_1^\times) \cong (\mathbb{P}^3 \times T^\times, D^\times) \cong (\mathcal{X}_2^\times, D_2^\times)\), and \((K_{X_1} + \frac{4}{d}D_1)|_{\mathcal{X}_1^\times} \sim 0\), the divisor \(K_{X_2} + \frac{4}{d}D_2\) vanishes on \(\mathcal{X}_2^\times\), hence by Lemma 3.6, \(dK_{X_2} + 4D_2 \sim 0\).

Next, run a \(K_{X_2}\) MMP over \(T\). This ends in a fibration \((\mathcal{X}, D)/T\), and \(\pi : (\mathcal{X}, D) \to (\mathbb{P}^3 \times T, D)\) is the required completion of \((\mathbb{P}^3 \times T^\times, D^\times)\), as we verify below. First, because it is the total space of an MMP fibration and the general fiber is \(\mathbb{P}^3\), \(\mathcal{X}/T\) is a Mori fiber space, \(\mathcal{X}/X\) is dlt, and \(\mathcal{X}\) is \(\mathbb{Q}\)-factorial. This implies that \(K_{\mathcal{X}}\) and \(D\) are \(\mathbb{Q}\)-Cartier. Also, because \(\rho(\mathcal{X}/T) = 1\), from the exact sequence used in Lemma 3.6, tensoring with \(\mathbb{Q}\) implies \(X\) is irreducible and therefore normal and log terminal [KM98, Proposition 5.51]. Finally, because \((\mathcal{X}_2, X_2 + \frac{4}{d}D_2)\) was dlt and \(dK_{X_2} + 4D_2 \sim 0\), we have that \((\mathcal{X}, X + \frac{4}{d}D)\) is log canonical and \(dK_{\mathcal{X}} + \frac{4}{d}D \sim 0\). Then, by adjunction, \((\mathcal{X}, \frac{4}{d}D)\) is log canonical and \(dK_{\mathcal{X}} + 4D \sim 0\).

Ultimately, this shows that families of log smooth pairs \((\mathbb{P}^3, D)\) over a punctured curve can be completed to a ‘well behaved one’: we can find a normal variety \(\mathcal{X}\) and a divisor \(D\) such that \((\mathcal{X}, \frac{4}{d}D_\mathcal{X})\) is log canonical that complete the family. However, there may be more than one such limit, even with the requirement that the canonical divisor of the family is \(\mathbb{Q}\)-Cartier. The problem arises precisely with pairs \((\mathbb{P}^3, D)\) such that the semistable limit \((\mathcal{X}, \frac{4}{d}D_\mathcal{X})\) is strictly log canonical. However, if we instead work with \(H\)-\(\epsilon\) stable pairs (Definition 3.3), we can modify these semistable limits to a unique limit, although we have to sacrifice normality of \(X\).

Next, we will prove that families of \(H\)-\(\epsilon\) stable pairs over a punctured curve can be extended (possibly after base change) in a unique way. Again, see Remark 3.10 for a summary of this process.

Theorem 3.9. Let \(0 \in T\) be the germ of a curve and write \(T^\times = T - 0\). Let \(D^\times \subset \mathbb{P}^3 \times T^\times\) be a family of smooth hypersurfaces over \(T^\times\) of degree \(d \geq 4\). Then, there exists a finite surjective base change \(T' \to T\) and a family \((X, D)/T'\) of \(H\)-\(\epsilon\) stable pairs extending the pullback of the family \((\mathbb{P}^3 \times T^\times, D^\times)/T^\times\) such that the divisors \(K_X\) and \(D\) are \(\mathbb{Q}\)-Cartier. The family is unique in the following sense: any two such families become isomorphic after a further finite surjective base change.

Proof. As constructed in the proof of Theorem 3.8, let \((X_1, D_1)\) be a family of \(H\)-\(\epsilon\) semistable pairs extending \((\mathbb{P}^3 \times T^\times, D^\times)\). By construction, \((X_1, X_1 + \frac{4}{d}D_1)\) is log canonical and the pair \((X_1, X_1)\) is dlt, as verified in the proof of Theorem 3.8. We can find a minimal dlt model \(\pi : (X_2, D_2) \to (X_1, D_1)\)
such that $dK_{X_2} + 4D_2 = \pi^*(dK_{X_1} + 4D_1)$ and $(X_2, X_2 + \frac{4}{3} + \epsilon)D_2$ is dlt [KK10, Theorem 3.1]. Hence, $(X_2, X_2 + \frac{4}{3} + \epsilon)D_2$ is dlt for $\epsilon$ sufficiently small [KM98, Corollary 2.39].

Now, let $(\mathcal{X}, \mathcal{D})$ be the $K_{X_2} + \frac{4}{3} + \epsilon\mathcal{D}_2$ canonical model. Then, $(\mathcal{X}, X + \frac{4}{3} + \epsilon\mathcal{D})$ is log canonical, $K_X + \frac{4}{3} + \epsilon\mathcal{D}$ is $\mathbb{Q}$-Cartier, $dK_X + 4\mathcal{D} \sim 0$, and $K_X + X + \frac{4}{3} + \epsilon\mathcal{D}$ is relatively ample. Therefore, by adjunction, $(X, \frac{4}{3} + \epsilon\mathcal{D})$ is slc, $dK_X + 4D \sim 0$, and $K_X + \frac{4}{3} + \epsilon\mathcal{D}$ is ample.

Further note that $K_X$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier, since $K_X + \frac{4}{3} + \epsilon\mathcal{D}$ is $\mathbb{Q}$-Cartier and $dK_X + 4\mathcal{D} \sim 0$, hence Cartier.

Finally, to see uniqueness, observe that any two such families have a common semistable log resolution (possibly after base change, suppressed in the notation). Then, because each family is slc and the divisor $K_X + \frac{4}{3} + \epsilon\mathcal{D}$ is ample for all $\epsilon$ sufficiently small, each family is a log canonical model of the resolution. Because log canonical models are unique, this implies (possibly after base change), the limits are unique. □

Remark 3.10. The diagram below summarizes the proof of properness (existence of unique limits).

\[
\begin{array}{c}
(\mathbb{P}^3 \times T^\times, \mathcal{D}^\times)_{T^\times} \xrightarrow{(1)} (\mathcal{X}, \mathcal{D}_\mathcal{X})_T \\
\downarrow \text{resolution} \\
(\mathcal{Y}, \mathcal{D}_\mathcal{Y}) \\
\downarrow \text{MMP} \\
(\mathcal{Y}', \mathcal{D}_\mathcal{Y}') \\
\downarrow \text{K}_{\mathcal{Y}'} \text{MMP} \\
(\mathcal{Y}, \mathcal{D}_\mathcal{Y}) \\
\downarrow \text{dlt mod.} \\
(\mathcal{Z}, \mathcal{D}_\mathcal{Z}) \\
\downarrow \text{K}_Z + (\frac{4}{3} + \epsilon)\mathcal{D} \text{ can. mod.} \\
(\mathcal{Z}, \mathcal{D}_\mathcal{Z})
\end{array}
\]

(1) Complete the family of pairs over a punctured curve.
(2) After base change (suppressed in the notation/diagram), there exists a semistable log resolution, changing only the central fiber.
(3) Given the resolution, run a $K_{Y} + \frac{4}{3}\mathcal{D}_Y$ MMP over $T$.
(4) Next, run a $K_{\mathcal{Y}'}$ MMP over $T$. This additional minimal model program ensures the special fiber is irreducible and both $K$ and $\mathcal{D}$ are $\mathbb{Q}$-Cartier.
(5) Take a minimal dlt modification [KK10, Definition 1.9] of $(\mathcal{Y}, \frac{4}{3}\mathcal{D}_\mathcal{Y})$, extracting any $-1$ divisors over the strictly log canonical locus. Because $(\mathcal{Z}, \frac{4}{3}\mathcal{D}_Z)$ is dlt and the singularities behave in a particular way, for $\epsilon$ sufficiently small, $(\mathcal{Z}, (\frac{4}{3} + \epsilon)\mathcal{D}_Z)$ is dlt.
(6) Finally, take the $K_Z + (\frac{4}{3} + \epsilon)\mathcal{D}_Z$ canonical model. Uniqueness follows.

4. Classification

There are many other ingredients in the study of these moduli spaces. For fixed $\epsilon$, the family of $\mathcal{H}$ stable pairs is bounded [HMX14b, Theorem 1.1], so we can embed all $\mathcal{H}$ stable pairs into a large projective space and (hope to) use the Hilbert scheme to construct a moduli space. However, we do not know the families are bounded as $\epsilon$ tends to 0. Therefore, it is of interest to try to bound the families of $\mathcal{H}$ stable pairs in another way. In Hacking’s work, there is a more elementary way
to show boundedness in terms of the degree $d$ as in [Hac04, Theorem 4.5]. However, this uses the existing classification of slc surfaces, so the method of proof does not generalize for threefolds.

As discussed in the introduction, we first classify the threefolds appearing in $H$-stable pairs and then obtain boundedness as a consequence. We dedicate our attention only to the threefolds $X$, not the pair $(X, D)$, since the ample divisor $D$ must be in a linear system determined by a multiple of $K_X$.

Classification is of interest for many reasons. As discussed in the introduction, if the varieties $X$ in the moduli problem are at worst semi-log terminal, boundedness is known by a result of Hacon, McKernan, and Xu [HMX14a, Corollary 1.7]. The condition on singularities and the fact that $dK_X + 4D \sim 0$ imply that these pairs are actually $\epsilon$-log terminal (meaning the discrepancy is greater than or equal to $-1 + \epsilon$ for some fixed $\epsilon > 0$). Therefore, existing results apply and show that the moduli problem is bounded. There is also the following result of de Fernex and Fusi, summarized nicely in a paper by Totaro, that implies these log terminal varieties are rational.

**Theorem 4.1.** [Tot16] Rationality specializes in families of complex klt varieties of dimension at most 3.

Therefore, if $X$ is a log terminal degeneration of $\mathbb{P}^3$, it is rational.

A classification of rational, log terminal varieties that admit a smoothing to $\mathbb{P}^3$ is not currently known. One necessary criterion is that $(-K_X)^3 = 64$ (see below), so certainly not all rational Fano varieties are eligible. This is an open question that, if solved, would contribute to a complete classification result of all pairs on the boundary of the moduli space of $H$-stable pairs. In light of Theorem 4.21, this is the only case that remains. We should point out that such a classification is known in dimension 2 (log terminal surfaces that smooth to $\mathbb{P}^2$) [Man91].

**Proposition 4.2.** If $f : X \to C$ is a flat family of $n$-dimensional projective varieties over a pointed curve $0 \in C$. Assume that $K_{X/C}$ is Q-Cartier. Assume that the general fiber $X_t$ is smooth and the special fiber $X_0$ is normal. Then, $(K_{X_0})^n = (K_{X_t})^n$. In particular, if $X_t \cong \mathbb{P}^3$, $(K_{X_0})^3 = -64$.

**Proof.** Let $l$ be an integer such that $lK_X$ is Cartier. Then, for any $t \in C$, $\mathcal{O}_{X_t}(lK_{X_t}) \cong \omega_{X_t}^{[l]} \cong (\omega_{X_t}^{[l]})_t$. By definition, $(lK_{X_t})^n$ is the coefficient of $m_1 m_2 \ldots m_n$ in $\chi(X_t, \mathcal{O}_{X_t}((m_1 + m_2 + \cdots + m_n)lK_{X_t}))$. Because $f$ is flat, this polynomial is constant, so $(lK_{X_t})^n = l^n [K_{X_t}^n]$ is constant. Therefore, $K_{X_t}^n$ is constant, as desired.

**Remark 4.3.** The assumption that $K_X$ is Q-Cartier is essential; see, for example, [KM98, Example 7.61].

It is also relatively easy to construct a non-rational degeneration of $\mathbb{P}^3$, as shown by the following example. Any such example is at least log canonical, in light of Theorem 4.1.

**Example 4.4.** Given a projectively normal variety $V \subset \mathbb{P}^N$, there is a standard degeneration of of $V$ to a cone over its hyperplane section [KM98, 7.61]. Thus, taking the 4-uple embedding $\mathbb{P}^3 \hookrightarrow \mathbb{P}^4$, the general hyperplane section of the image corresponds to a K3 surface in $\mathbb{P}^3$, which has trivial canonical divisor. The cone $X$ over such a surface $S$ is log canonical: let $Y$ be the blow up of $X$ at the vertex, $f : Y \to X$. Then, $f$ is birational with exceptional divisor isomorphic to $S$, so $K_Y \sim f^* K_X + aS$. By adjunction, $K_S \sim (K_Y + aS)|_S$, so $0 \sim K_S \sim (f^* K_X + (a + 1)S)|_S$. Given any curve $C \subset S$, $0 = K_S \cdot C = (a + 1)S|_C$, hence $a = -1$ and $X$ is log canonical by definition. A calculation shows that $-K_Y - S$ is nef and 0 exactly on curves contained in the exceptional locus $S$, so $-K_X$ is ample. However, for $X$ to occur as a threefold in a pair $(X, D)$ on the boundary of the moduli space above, we must have $-\frac{d}{4} K_X + D$. By the discussion above, $K_X \cdot C \in \mathbb{Z}$ for any curve $C \subset X$, and a calculation shows $K_X \cdot \Gamma = -1$ for a ruling of the cone. Since the singularity of $X$ is strictly log canonical, in order for $(X, (\frac{d}{4} + \epsilon)D)$ to also be log canonical, $D$ must miss the singularity of $X$. Hence, $D$ is contained in the smooth locus of $X$ and is therefore Cartier, so $D \cdot C \in \mathbb{Z}$, which implies $\frac{d}{4} \in \mathbb{Z}$. Therefore, for $d$ not divisible by 4, $X$ cannot occur as a boundary threefold.
From this observation and the comment on boundedness above, we first focus on the strictly log canonical threefolds appearing in the moduli problem. The main result is that, for odd degree $d$, there are none.

### 4.1. Strictly log canonical Fano threefolds

The inspiration for classification of the strictly log canonical threefolds in this moduli problem is the following theorem:

**Theorem 4.5.** [Ish91] If $X$ is a normal, Gorenstein variety of dimension $n$ with $K_X$ anti-ample and with finite (non-empty) irrational locus, then $X$ is a cone over a variety $S$ with canonical singularities and $K_S \sim 0$.

If $X$ is a normal, Gorenstein variety with $K_X$ anti-ample, the log canonical locus coincides with the irrational locus [KM98, Corollary 5.24]. Therefore, this theorem implies that if a threefold $X$ has a finite (non-empty) lc locus, it is either a cone over a K3 surface or two dimensional Abelian variety.

The following is a generalization of this result.

**Theorem 4.6.** Let $X$ be a projective variety with a finite number of strictly log canonical singularities $\{p_1, \ldots, p_n\}$ and $-K_X$ ample. If $a(E, X) \in \{-1, \mathbb{R}^{\geq 0}\}$ for every exceptional divisor $E$ over $X$ with center $X(E) \subset \{p_1, \ldots, p_n\}$, then $X$ is a cone over a variety $Z$ with $K_Z \equiv 0$.

The extra hypotheses in this result arise from removing the Gorenstein hypotheses in Theorem 4.5. In order to ensure $X$ is a cone, there needs to be a certain extremal ray in the cone of curves.

Before getting to the proof, we provide a few definitions and technical lemmas. In all cases, we consider dlt pairs $(X, D)$ and study properties of various $K_X$-negative and $K_X + D$-negative contractions. The motivating idea is to study contractions that happen ‘over’ $D$. Divisorial contractions that are $K_X + D$-negative and $D$-positive must have a certain structure, as explained below.

**Definition 4.7.** Given a proper variety $X$, the effective cone $NE(X)$ is the collection of effective 1 cycles on $X$ modulo numerical equivalence. We usually consider the closure $\overline{NE(X)}$.

**Definition 4.8.** If $R$ is an extremal ray in $\overline{NE(X)}$, we say that the contraction of $R$ is an elementary extremal contraction. In what follows, we will refer to the contraction of $R$ as simply an extremal contraction and always mean the contraction of an extremal ray.

We begin by discussing the negativity of $K_X$ in certain $K_X$-negative contractions. Namely, the next lemma shows that $K_X$ cannot be ‘too’ negative on fibers.

**Lemma 4.9.** Let $X$ be a normal projective variety such that $K_X$ is $\mathbb{Q}$-Cartier. If $\phi : X \to Y$ is a contraction of a $K_X$-negative extremal ray with fibers of dimension at most 1, then each fiber $F$ is a chain of $\mathbb{P}^1$’s whose configuration is a tree such that $-1 \leq K_X : C < 0$ for each irreducible component $C$ of $F$.

**Proof.** By assumption, $R^2 \phi_* F = 0$ for any coherent sheaf $F$ on $X$. By Grauert-Riemenschneider vanishing, $R^1 \phi_* \omega_X = 0$, and by [KMM87], because $-K_X$ is $\phi$-ample, $R^1 \phi_* O_X = 0$. Then, consider any sheaf of ideals $J$ such that $O_X/J$ is supported on a fiber $F$ of $\phi$:

$$0 \to J \to O_X \to O_X/J \to 0$$

Pushing forward to $Y$, we see that $R^1 \phi_* O_X/J = H^1(F, O_X/J|_F) = 0$. Similarly, $H^1(F, \omega_X/J\omega_X|_F) = 0$ so $H^1(F, (\omega_X/J\omega_X)|_F/T) = 0$, where $T \subset (\omega_X/J\omega_X)|_F$ denotes torsion. Taking $J$ to be the ideal of $F$, we see that $F$ is a chain of $\mathbb{P}^1$’s whose configuration is a tree. Then, consider an irreducible component $C \subset F$ and the sheaf $\omega_X \otimes O_C/T$, where $T$ is the torsion in $\omega_X \otimes O_C$. This is a torsion-free sheaf on $\mathbb{P}^1$, so must be a vector bundle of the form $\omega_X \otimes O_C/T \cong \oplus O_{\mathbb{P}^1}(a_i)$. The vanishing of $H^1$ given above implies that $a_i \geq -1$ for each $i$. If $m$ is an integer such that $\omega_X^m$ is Cartier, we must have that $\omega_X^m \otimes O_C = O_{\mathbb{P}^1}(b)$ is a negative degree line bundle. But, there is a nonzero morphism $(\omega_X \otimes O_C/T)^{\otimes m} \to \omega_X^m \otimes O_C$, and $a_i \geq -1$ implies that $b \geq -1$. Therefore, $-1 \leq K_X : C < 0$. □
The previous lemma bounds the negativity of $K_X$. Morally, this hints that if $K_X \cdot C \geq -1$ for contracted curves, and $C \cap D \neq \emptyset$, that should force $(K_X + D) \cdot C \geq 0$. Certainly this could be false if $X$ was highly singular and $D \cdot C \notin \mathbb{Z}$, but with a few restrictions on the singularities, we can apply the lemma to our advantage.

We begin with an observation about these contractions.

**Lemma 4.10.** If $(X, D)$ is dlt and $D$ is an effective prime divisor that is Cartier in codimension 2, then any $K_X + D$-negative extremal divisorial contraction is an isomorphism on $D$ if and only if the exceptional divisor does not intersect $D$.

**Proof.** Let $\phi : X \to Y$ be the given contraction. Because $\phi$ is $K_X + D$ negative and divisorial, the negativity lemma implies

$$\phi^*(K_Y + D') = K_X + D - aE$$

where $D' = \phi_* D$, $E$ is the exceptional divisor, and $a > 0$. Because $D$ is Cartier in codimension 2, $K_X + D|_D = K_D$, so restricting this to $D$ gives

$$\phi^*(K_D + Diff_{D'}(0)) = K_D - aE|_D$$

where $Diff_{D'}(0)$ is the correction term to the adjunction formula needed if $D'$ is not Cartier in codimension 2. This correction term is effective, and $aE|_D$ is antieffective, so $\phi|_D : D \to D'$ is an isomorphism if and only if $E|_D = 0$. This also shows that $Diff_{D'}(0) = 0$. \hfill \Box

From this observation and Lemma 4.9, we can draw a number of corollaries. Namely, if we have a ‘nice’ contraction that is an isomorphism on $D$, because the fibers are well behaved, this will force the map to be a fibration.

However, one should be cautious; this lemma (and the corollaries) are false without the hypothesis that $D$ is Cartier in codimension 2.

**Example 4.11.** Let $X = \mathbb{P}^2$ and let $\pi : Y \to X$ be the $(n, 1)$ weighted blow up of the point $(0, 0)$ in linear coordinates $(x/z, y/z)$ for any $n > 1$. Let $L = (y = 0)$ be a line in $\mathbb{P}^2$ and let $L_Y$ be the strict transform. Denote the exceptional divisor of $\pi$ by $E$ and note that $E^2 = -\frac{1}{n}$. By construction, $L_Y$ and $E$ intersect at the unique $\frac{1}{n}(1, n - 1)$ singularity of $Y$ and are not Cartier at that point. We can compute

$$\pi^*K_X = K_Y - nE$$

and

$$\pi^*L = L_Y + E$$

so that $K_Y \cdot E = -1$ and $L_Y \cdot E = \frac{1}{n}$. Then,

$$\pi^*(K_X + L) = K_Y + L_Y - (n - 1)E$$

so the contraction $\pi : Y \to X$ of $E$ is $K_Y + L_Y$-negative and is an isomorphism on $L_Y$, but $L_Y \cap E \neq \emptyset$.

If $D$ is Cartier in codimension 2, however, we avoid the behavior in the previous example.

**Corollary 4.12.** If $(X, D)$ is dlt and $D$ is an effective, prime divisor that is Cartier in codimension 2, then any $K_X + D$-negative, $D$-positive extremal contraction that contracts a divisor but contracts no curves in $D$ is a Fano fiber contraction $X \to D$.

**Proof.** Let $\pi : X \to Y$ be the contraction. If a divisor is contracted, then the morphism is either a divisorial contraction or Fano fiber contraction onto a variety with strictly lower dimension. If no curves in $D$ are contracted, the induced map $D \to \pi(D)$ is finite, but $(Y, \pi_* D)$ is dlt by [KM98], hence $\pi_* D$ is normal. Furthermore, because no curves in $D$ are contracted, the fibers have dimension at most 1. But, if $\pi$ was divisorial, Lemma 4.9 implies $K_X \cdot C \geq -1$ for $C$ contracted by $\pi$. However, because $D$ is Cartier in codimension 2, for a general fiber $C$, $D \cdot C \in \mathbb{Z}$, hence $(K_X + D) \cdot C \geq 0$. 

Therefore, the contraction must be a fibration with general fiber $\mathbb{P}^1$. In this case, for general fiber $C$, $K_X \cdot C = -2$, so we must have $D \cdot C = 1$, so $\pi|_D : D \to \pi(D)$ is generically of degree 1. Therefore, by Zariski’s Main Theorem, and because $D$ is prime, $\pi_* D$ must be isomorphic to $D$ and $\pi : X \to Y$ is a Fano fiber contraction and $Y \cong D$. In particular, $X$ is almost a $\mathbb{P}^1$-bundle over $D$ (the general fiber is $\mathbb{P}^1$) and the fiber $D$ is a section of this almost-bundle. \hfill $\square$

**Corollary 4.13.** If $X$ is a terminal variety and $(X,D)$ is dlt for some effective prime divisor $D$ where $-D|_D$ is nef, then any $K_X+D$-negative $D$-positive contraction gives a Fano fibration $X \to D$.

*Proof.* If $X$ is terminal, the singular set has codimension at least 3 in $X$, hence $D$ is Cartier in codimension 2. If $-D|_D$ is nef, then any $D$-positive contraction contracts no curves in $D$, so by Corollary 4.12, the contraction of such a ray gives $X$ the structure of an almost-$\mathbb{P}^1$-bundle over $D$, or precisely, a Fano fibration $X \to D$. \hfill $\square$

We should point out that Lemma 4.9 does not require the contraction be divisorial; it could be a small contraction and the result would still hold. Although small contractions behave remarkably differently than divisorial contractions, we can still ask about small contractions that enjoy many of the same properties as those above. In particular, the next lemma shows that small contractions with $K_X+D$-negative and $D$-positive properties cannot exist.

**Lemma 4.14.** If $X$ is a terminal variety and $(X,D)$ is a canonical pair such that $D$ is an effective prime divisor, then the contraction of a $K_X+D$-negative, $D$-positive extremal ray $R$ that contracts no curves in $D$ cannot be a small contraction.

*Proof.* Assume the small contraction exists. Because this is a $K_X$-negative contraction, we consider the flip of $\phi$ as in the following diagram, where $Z$ is the blow up of the exceptional locus.

\[
\begin{array}{ccc}
Z & \xrightarrow{\pi} & X^+ \\
\downarrow & & \downarrow \phi^+ \\
Y & \xleftarrow{\phi} & X
\end{array}
\]

Note that the fiber of the contraction $\phi : X \to Y$ is not contained in $D$, by assumption. Because every exceptional divisor $E$ has nonnegative discrepancy $a(E,X,D)$, if $D_Z = \pi_*^{-1}D$, we have

\[K_Z + D_Z = \pi^*(K_X + D) + \sum a_i E_i\]

where $a_i \geq 0$ for each $i$. Restricting to $D$, because $D$ is Cartier in codimension 2, we get

\[K_{D_Z} = \pi|_D^*(K_D) + \sum a_i E_i|_{D_Z} \cdot 
\]

But, by Lemma 3.38 in [KM98], flips can only improve singularities, so

\[\pi^+(K_X^+ + D^+) = \pi^+(K_X + D) - \sum c_i E_i\]

where $c_i \geq 0$. Then, restricting to $D$ we see that

\[\pi^+(K_{D^+} + D |_{D^+}(0)) = \pi^+(K_D) - \sum c_i E_i|_{D_Z} \cdot 
\]

Substituting, we see that

\[\pi^+(K_{D^+} + D |_{D^+}(0)) = K_{D_Z} - \sum a_i E_i|_{D_Z} - \sum c_i E_i|_{D_Z} \cdot 
\]

However, $\pi|_{D_Z}$ was the resolution of the rational map $D \dasharrow D^+$, so $\pi^+|_{D_Z} : D_Z \to D^+$ is an isomorphism, which is a contradiction. \hfill $\square$
We can tie the previous lemmas together in the following result, seemingly technical but the key idea in the proof of Theorem 4.6.

**Lemma 4.15.** Let $X$ be a terminal variety and $(X, D)$ a canonical pair with $D$ an effective integral divisor such that $K_X|D$ is nef. If the class of a $K_X + D$-negative extremal ray $R$ contains a curve $C$ such that $C \cap D$ is finite and non-empty, and the contraction of $R$ has fiber dimension at most 1, then it must be a Fano fibration $X \to Y$ such that the general fiber is isomorphic to $\mathbb{P}^1$ and $Y \cong D$.

**Proof.** Because terminal varieties are singular only in codimension $\geq 3$, the assumption on $(X, D)$ implies that $D$ is Cartier in codimension 2.

By Lemma 4.14, the contraction of $R$ cannot be a small contraction. However, any curve $C \subset D$ with $K_X \cdot C \geq 0$, hence the contraction $\phi: X \to Y$ of a $K_X + D$-negative $D$-positive extremal ray cannot contract any curves in $D$. Therefore, $\phi|_D : D \to \phi(D)$ is a finite morphism. Note that, for general $C$ contracted by $\phi$, $D \cdot C > 0$ and $D \cdot C \in \mathbb{Z}$ because $D$ is Cartier in codimension 2. Assume for contradiction that $\phi$ was a divisorial contraction. Then, because no curves in $D$ are contracted, the fibers of $\phi$ have dimension at most one. If this is the case, Lemma 4.9 implies that $K_X \cdot C \geq -1$. However, this means $(K_X + D) \cdot C \geq 0$, contradicting our assumption.

Therefore, we must have $\phi: X \to Y$ a Fano fiber contraction of relative dimension 1. This implies the general fiber of $\phi$ is isomorphic to $\mathbb{P}^1$, as desired. To see that $Y \cong D$, note that the map $\phi|_D : D \to \phi(D)$ is finite but, for general fiber $C$ of $\phi$, $K_X \cdot C = -2$, so in order for $\phi$ to have been a $K_X + D$ negative contraction, we must have $D \cdot C = 1$. Therefore, $\phi|_D$ is finite and generically of degree 1, so by Zariski’s Main Theorem, $\phi|_D : D \to \phi(D) = Y$ is an isomorphism.

We are now ready to prove Theorem 4.6.

**Proof.** Taking a minimal $\mathbb{Q}$-factorial dlt model of $X$, there is a $\mathbb{Q}$-factorial variety $Y$ and a morphism $\pi : Y \to X$ extracting all divisors $E_i$ with discrepancy $a(E_i, X) = -1$ such that $K_Y$ is relatively nef. Let $E = \sum E_i$ and observe that $K_Y + E = \pi^*K_X$. Because $-K_X$ is ample and $K_Y + E$ is trivial on $E$ and negative on all curves not contained in $E$, there must exist a $K_Y + E$ negative, $E$ positive extremal ray $R$ in $\overline{NE(Y)}$.

Let $\phi : Y \to S$ be the contraction of $R$. By assumption, the pair $(Y, E)$ is canonical along $E$ (since $a(F, Y, E) = a(F, X)$ for any exceptional divisor $F$ over $X$), so Lemma 4.12 applies and $\phi : Y \to S$ is a fiber contraction of relative dimension 1. Note that, for a general fiber $F$ of $\phi$, $K_Y \cdot F = -2$ and $[F] \in R$, so $(K_Y + E) \cdot F < 0$. Choosing an appropriate fiber that misses the singular points of $Y$, one sees that $E_i \cdot F \in \mathbb{Z}$ for each $i$ because $F$ is contained in the smooth locus of $Y$. Therefore, because $K_Y \cdot F = -2$ and $(K_Y + E) \cdot F < 0$, there is only one exceptional divisor $E_0 = E$. Because $(Y, E)$ is dlt, $E$ is normal and $\phi$ contracts no curves in $E$, hence $S \cong E$, giving $\phi : Y \to S$ the structure of a $\mathbb{P}^1$ bundle. However, $E$ is contractible $\pi : Y \to X$, we see that $X$ is cone over $E$ (where ‘cone’ is interpreted as the contraction of a section of a $\mathbb{P}^1$-bundle over $E$). We can further characterize $E$ by observing that $(K_Y + E)|_E = K_E$, hence $K_E$ is numerically trivial.

Since one cannot guarantee that the exceptional divisors over a variety are in the set given in Theorem 4.6, we first make an easy observation, whose proof is the same as that above.

**Proposition 4.16.** Let $X$ be a projective variety with a finite number of strictly log canonical singularities $\{p_1, \ldots, p_n\}$ and $-K_X$ ample. Consider a minimal dlt modification $\pi : Y \to X$ extracting the $-1$ divisors of $X$, so $K_Y + E = \pi^*(K_X)$. If there exists an extremal ray $R \in \overline{NE(Y)}$ such that a curve $C \not\subset E$, $[C] \in R$, intersects $E$ at a smooth point of $Y$, then $X$ is a cone over a numerically Calabi-Yau variety.

To remove the restrictions on the discrepancies in Theorem 4.6, we would like to say there always exists a ray as in Proposition 4.16. However, it is not obvious why this is true or clear that it should be true. Instead, we include various generalizations of the result Theorem 4.6.
Finally, if a fibration: we can never change the general curve from a K with a Fano variety, if we run a standard minimal model program, it should terminate in a Fano variety. Because we are starting with a non-rational variety, it is for surfaces, there are easy examples of log canonical singularities of log canonical singularities where an exceptional divisor is not rational or ruled, related to the fact that log canonical singularities do not have to be rational singularities. So, one might expect that -1 exceptional divisors over a log canonical singularity are often not rational or ruled. If that is the case, the following result characterizes these singularities.

**Theorem 4.17.** If X is a projective 3-dimensional variety with a finite number of strictly log canonical singularities and -K_X ample such that at least one exceptional divisor E over X with discrepancy a(E, X) = -1 is not rational or ruled, then there is only one such E and X is birational to a P^1 bundle over E.

**Proof.** We proceed in a similar fashion to that of the previous proof.

There is a Q-factorial variety Y and a morphism π : Y → X extracting all divisors Δ_i with discrepancy a(Δ_i, X) ≤ 0 such that K_Y is relatively nef. Let E = \( \sum \Delta_j \) be the sum over divisors Δ_j with discrepancy -1 and \( F = \sum -a(\Delta_k, X)\Delta_k \) be the sum over divisors with discrepancy larger than -1. By construction of Y (which is terminal, hence has finitely many singular points), these effective divisors are Cartier in codimension 2, \( π^*(K_X) = K_Y + E + F \), and for any curves \( C \subset Supp(E + F) \), \( K_Y \cdot C \geq 0 \). By assumption on X, the general curve through E has negative K_X-degree.

We would like to find an E positive and K_Y + E negative extremal ray in the cone of curves. If so, we can proceed as in the proof of Theorem 4.6 to conclude that the contraction of such a ray gives a Fano fibration \( φ : Y → E \) (and E consists of only one component). Because a general curve through E doesn’t intersect F, \( F = 0 \). Therefore, the log canonical locus in X consists of a single point \( x \in X \), and for a general fiber l of \( φ, π^*K_X \cdot l = -1 \) and \( E \cdot l = 1 \).

In general, by the construction of Y, there must exist \( K_Y + E + F \)-negative and \( E + F \)-positive extremal rays. If we cannot find a ray that is \( E \)-positive, because \( K_Y \) is nef relative to \( π : Y → X \), we must have every \( K_Y \)-negative and \( K_Y + E \)-negative extremal ray be \( E \)-trivial. We proceed by running an MMP on Y, contracting \( K_Y \)-negative rays. (Note this is an MMP on Y, not on the pair \( (Y, E + F) \).) At any point in time, if we reach an intermediate variety \( Y' \) with a \( K_{Y'} + E' \)-negative \( E \)-positive extremal ray \( R \) in \( NE(X) \), the MMP terminates with the contraction of \( R \) if the fiber dimension is at most 1. This follows from Lemma 4.10 and Lemma 4.14.

Assume we do not find such a ray. Because X was a Fano threefold, the MMP must terminate with a Fano fibration \( f : Y' → S \) such that \( \dim S < 3 \). We claim that the only components of \( E \) that could be contracted by an MMP are rational or ruled. If \( φ : Y' → Y'' \) is a divisorial contraction of a component \( Δ \) of \( E' \), because \( (Y, E) \) is canonical (and hence \( (Y', E') \) is canonical), \( Δ \) is a canonical, rationally connected surface. Because such surfaces are rational, the result follows. If no component of \( Δ \) is contracted until the termination of the MMP \( f : Y' → S \), there are a few cases to consider. Either \( Y' \) is a terminal Fano variety of Picard rank 1, and because \( K_{Y'} + Δ \) is negative, \( Δ \) is a smooth Fano surface, hence rational. If \( Y' \) has Picard rank 2 and \( C \) is a curve, if \( Δ \) is a fiber of \( f' \), again it is smooth, Fano, and rational. If instead \( f'|_Δ : Δ → C \) is surjective, \( Δ \) is a ruled surface. Finally, if \( \dim S = 2 \), \( \dim f'(Δ) = 0 \) implies \( Δ \) is Fano and therefore rational. If \( \dim f'(Δ) = 1 \), \( Δ \) is again a smooth ruled surface, and we are left only with the desired result, \( S \cong Δ \).

Note this implies that \( E \) has at most one non-rational or ruled component.

Note that, for a log canonical variety, there is certainly no need for such a non-rational or ruled exceptional divisor to exist in the resolution. In fact, even for surfaces, there are easy examples of log canonical singularities whose resolution graphs consist only of rational curves. For classification purposes, we would like to also characterize these log canonical threefolds. Because we are starting with a Fano variety, if we run a standard minimal model program, it should terminate in a Fano fibration: we can never change the general curve from a K-negative curve to a K-non-negative curve. So, taking a modification \( X' → X \) extracting the -1-divisors, a run of the MMP on \( X' \) will terminate in a fibration \( X'' → Z \), where \( Z \) has dimension 0, 1, or 2. In the study of moduli of...
pairs \((X,D)\) where these varieties appear as \(X\), we would like to understand the structure of the fibration \(X'' \to Z\). In particular, in Section 5, an understanding of these fibrations will illuminate the requirement that \(d\) be odd in Theorem 1.3. Therefore, it will be beneficial to understand the the termination of the MMP in these cases, which is the content of the following result.

**Theorem 4.18.** If \(X\) is a strictly log canonical threefold such that \(-K_X\) is ample, \(dK_X + 4D \sim 0\) for some prime \(\mathbb{Q}\)-Cartier Weil divisor \(D\), and \(D\) does not contain the locus of strictly log canonical singularities, then \(d\) is even.

**Proof.** This is clear in the settings mentioned above: where \(X\) is Gorenstein, the discrepancies of \(X\) are in the set \(\{-1, \mathbb{R}^{\geq 0}\}\), or there is a non-rational or ruled component of \(E\). More generally, it is true in the following setting:

Consider a \(\mathbb{Q}\)-factorial dlt modification of \(X\): a morphism \(\pi: X' \to X\) extracting the \(-1\) divisors over the strictly log canonical singular points. Note that \(K_{X'} + E = \pi^*K_X\).

Running a \(K_{X'}\) minimal model program, if this terminates in a fibration \(f: X'' \to S\) of relative dimension 1 or 2, we find that \(d\) must be even.

In the first case, we can choose a ruling \(l'\) of \(X\) (an image of one of the \(\mathbb{P}^1\)'s \(l\) contracted in the fibration \(f: X'' \to S\) on \(X\)) sufficiently generally so \(l'\) intersects \(D\) where \(D\) is a Cartier divisor. Then, \(D \cdot l' \in \mathbb{Z}\), but \(D \cdot l' \equiv -\frac{d}{2}K_X \cdot l'.\) If \(E\) is not ample with respect to \(f\), for sufficiently generic \(l\), we find that \(E \cdot l = 0\), hence \((K_{X''} + E) \cdot l = -2\) so \(K_X \cdot l' = -2\). If \(E\) is ample with respect to \(f\), because the generic curve on \(X'\) was \(K_{X'} + E\)-negative, we must have \(E \cdot l = 1\) so \((K_{X''} + E) \cdot l = -1\) so \(K_X \cdot l' = -1\). In either case, because \(D \cdot l' \equiv -\frac{d}{2}K_X \cdot l'\), we must have \(d\) even.

We make a similar argument if \(f: X'' \to S\) is a fibration of relative dimension 2. Because the general fiber is a smooth Fano surface \(L\), this implies that there exist curves \(l\) in a general fiber with \(K_{X''} \cdot l = K_L \cdot l = -2\) or \(-3\). Because \(E \cdot l \geq 0\) and \((K_{X''} + E) \cdot l < 0\), we have that \(K_X \cdot l' = -1, -2, \) or \(-3\). In any case, choosing \(l\) generally so \(l'\) intersects \(D\) where it is Cartier implies that \(4|d\) or \(2|d\).

To complete the proof, it suffices to show that \(d\) is even if all runs of the minimal model program on \(X'\) terminate in a Fano threefold \(X''\) with \(\rho(X'') = 1\).

Assume for contradiction that this was the case. Then, we must have contracted divisors \(D\) over \(E\) such that \(E \cap D \subset \text{Sing } X\). If we contracted any divisor \(D\) such that the generic point of intersection \(E \cap D\) is smooth, then by Lemma 4.10, the minimal model program would have terminated in a fibration. Furthermore, we must have contracted at least divisor \(D\) to a curve in \(E_i\) over each component \(E_i\) of \(E\). Because the fibers of \(\pi: X' \to X\) are \(E\)-negative and \(K\)-positive, in order to terminate with \(\rho(X'') = 1\), we must have either contracted each component of \(E\) or turned \(E\) into an ample divisor. In either case, all curves in \(E\) contracted by \(\pi\) must change sign with respect to \(K\), hence there must have been contractions of divisors \(D\) intersecting those curves.

Therefore, we can assume that \(X\) is strictly canonical or worse along \(E\)-ample curves in \(E\) so that these contractions exist. Furthermore, any flip or contraction of a divisor to a point will not change the intersection theory for the general curve in \(E\), hence we will focus only on contractions of divisors to points.

With this in mind, we will start with a slightly different set-up and begin with a terminal variety \(Y\) instead of a dlt variety \(X'\). We will show directly that \(d\) must be even by finding a minimal model of \(Y\).

First, as in previous proofs, we find the desired \(\mathbb{Q}\)-factorial variety \(Y\) and a morphism \(\pi: Y \to X\) extracting all divisors \(\Delta_i\) with discrepancy \(a(\Delta_i, X) \leq 0\) such that \(K_Y\) is relatively nef and \(\pi\) factors through \(X'\). Let \(E = \sum \Delta_j\) be the sum over divisors \(\Delta_j\) with discrepancy \(-1\) and \(F = \sum -a(\Delta_k, X)\Delta_k\) be the sum over divisors with discrepancy larger than \(-1\). By construction of \(Y\) (which is terminal, hence has finitely many singular points), these effective divisors are Cartier in codimension 2, \(\pi^*(K_X) = K_Y + E + F\), and for any curves \(C \subset \text{Supp}(E + F), K_Y \cdot C \geq 0\). By assumption on \(X\), the general curve through \(E\) has negative \(K_X\)-degree.
We begin by contracting a $K_Y$-negative, $E + F$ positive, $K_Y + E + F$ negative ray on $Y$. In order for $X''$ to exist as above, eventually curves in $E$ must change sign so we will focus only on divisorial contractions such that the image of the divisor is a curve on $E$. If a divisor $D$ over $E$ was contracted such that $D$ was not contained in $	ext{Supp} F$ and $E$ intersected some curve in a smooth point, we claim that the minimal model program would have terminated in a fibration.

Indeed, if the contraction of $D$ was finite over $E$, this follows from Lemma 4.9. If the contraction of $D$ contracted a curve in $E$, this follows from [BHN15, Lemma 2.1] and its application to the proof of [BHN15, Theorem 1.3].

Then, take the image of that divisor in $X'$ to obtain a contradiction. Hence, for curves in $E$ to change sign, we must contract components of $\text{Supp} F$ onto curves in $E$. However, the fibers of $\pi : Y \to X$ are non-positive with respect to $K$, so to contract a component $\Delta$ of $\text{Supp} F$ in this way, we must first have performed divisorial contractions over $\Delta$ such that the image of the contracted divisor is a curve on $\Delta$.

Therefore, we will assume the only $K_Y$-negative, $E + F$ positive, $K_Y + E + F$ negative rays on $Y$ are $E$-trivial and contracting them is a divisorial contraction $\phi : Y \to Y'$ such that $\phi$ contracts a divisor $\Gamma$ and $\phi(\Gamma)$ is a curve on $\Delta \subset \text{Supp} F$.

We know $\text{Supp} F$ is the preimage under $\pi'$ of the locus of strictly canonical or worse singularities on $(X', E)$ where $\pi' : Y \to X'$. Furthermore, by the discussion above, we are only interested in components $\Delta$ of $\text{Supp} F$ mapping to a curve in $E$ under $\pi'$. In this case, consider the pullback

$$K_Y + E + a_j \Delta_j = \pi^*(K_{X'} + E)$$

where, by abuse of notation, $E$ denotes itself and its strict transform. Restricting to $E$, because $Y$ is terminal, we find

$$K_E + a_j \Delta_j = \pi^*(K_E + \text{Diff}_E)$$

where $\text{Diff}_E$ is the correction term needed in the adjunction formula. By [Kol13, Remark 4.4], the coefficients of the different are 1 or $1 - \frac{1}{m}$. Our assumption on $(X', E)$ implies that all coefficients are less than 1, so for the components $\Delta_j$ whose image on $X'$ is a curve in $E$, the coefficients $a_j$ are of the form $1 - \frac{1}{m_j}$ for some integer $m_j$.

Assume the contraction of $\Gamma$ intersects a component $\Delta_j$ with $m_j \geq 2$. In this case, we claim that the image $l'$ of the general fiber $l$ of $\phi|_{\text{Supp} F}$ on $X$ satisfies $K_X \cdot l' = \frac{1}{m_j}$. Indeed, if $l$ intersects only $\Delta_j$, by negativity of the contraction, we must have $\Delta_j \cdot l = 1$ and $K_Y \cdot l = -1$. Then for its image $l'$ on $X'$, $K_X \cdot l' = (K_{X'} + E) \cdot l' = (K_Y + E + \left(1 - \frac{1}{m_j}\right) \Delta_j) \cdot l = -\frac{1}{m_j}$ because $l'$ intersects only $\Delta_j$, for generic $l$, its image on $X$ intersects $D$ where $D$ is Cartier, hence $D \cdot l' \in \mathbb{Z}$. Therefore, from the relationship $dK_X + 4D \sim 0$, we find that $d$ is even.

If $l$ intersected a divisor $\Delta_k$ other than $\Delta_j$, then the contraction would not have been $K_Y + E + F$-negative. First, if $\Delta_k$ is another component contracted to a curve in $E$, this follows from the coefficient being $1 - \frac{1}{m_k}$. If $\Delta_k$ is a divisor whose image is not contained in $E$, this can be seen using only intersection theory: it would imply that the coefficient $a_k$ of $\Delta_k$ is $a_k < a_j = 1 - \frac{1}{m}$, however the existence of $K_Y$-negative and $\Delta_k$-trivial curves would force $a_k \geq a_j$.

Suppose for simplicity there are only two components of $\text{Supp} F$, $\Delta_1$ and $\Delta_2$, with $a_1 = 1 - \frac{1}{m}$. There must exist $K$-negative and $\Delta_2$-trivial curves. Assuming the contraction of $l$ was the only possible $K$-negative contraction, we know that $\Delta_1 \cdot l = 1$ and $\Delta_2 \cdot l = n \geq 1$. Then, we know $K_Y \cdot l = -1$, $E \cdot l = 0$, $\Delta_1 \cdot l = 1$, $\Delta_2 \cdot l = n$, and $D \cdot l = -1$. For the curves $C$ contracted by $\pi$ in $\Delta_2$, because $\Delta_2$ lies over a strictly log terminal locus of singularities in $X$, the fibers are rationally connected, hence we can assume $(K_Y + \Delta_2) \cdot C = -2$ or $-3$. Assume for the moment the intersection is $-2$ (the proof is the same if it is $-3$). Therefore, $K_Y \cdot C = a$, $E \cdot C = 0$, and $\Delta_2 \cdot C = -2 - a$. Let $D \cdot C = d$. A computation shows that $a_2 = \frac{a}{2+2a}$. In order for there to exist curves that are $\Delta_2$-trivial and $K_Y$-negative, we must have $nd \geq 2 + a$ and $na < 2 + a$. This implies that $n = 1$ or $n = 2$ and $a = 1$. 
In either case, we must have \( K + E + F \) negative on \( l \), so
\[
(K + E + F) \cdot l = \left( K + E + \left( 1 - \frac{1}{m} \right) \Delta_1 + \frac{a}{2 + a} \Delta_2 \right) \cdot l \leq 0
\]
but
\[
\left( K + E + \left( 1 - \frac{1}{m} \right) \Delta_1 + \frac{a}{2 + a} \Delta_2 \right) \cdot l = -1 + 0 + \left( 1 - \frac{1}{m} \right) + \frac{na}{2 + a} = \frac{na}{2 + a} - \frac{1}{m}.
\]

If \( n = 2 \) and \( a = 1 \), this is a contradiction. Similarly, if \( n = 1 \) and \( m > 2 \), this is a contradiction. For the finite number of exceptional cases, we investigate them by hand and and compute \( K_X \cdot \pi(l) \) and show that it always forces \( d \) to be even. \( \square \)

One might ask, taking the minimal model of a terminalization of our log canonical threefold, what fibrations can appear. Indeed, there exist examples of both relative dimension 1 and 2 occurring. For instance, the cone over a K3 surface, Example 4.4, is an example with one \(-1\) exceptional divisor \( E \) whose dlt model (which happens to be a resolution of singularities) is a \( \mathbb{P}^1 \) bundle over \( E \).

For examples of threefolds with log canonical singularities whose associated model is a fibration of relative dimension 2, many appear in [BHN15]. For convenience, we sketch [BHN15, Example 6.1] here. Details can be found in the original paper. Consider \( X' = \mathbb{P}(O_C^{\oplus 2} \oplus \mathcal{L}) \), where \( C \) is a smooth, genus 1 curve, and \( \mathcal{L} \) is an ample line bundle on \( C \). If \( E \cong C \times \mathbb{P}^1 \) is the divisor defined by the quotient \( O_C^{\oplus 2} \oplus \mathcal{L} \to O_C^{\oplus 2} \), then there is a birational morphism \( X' \to X \) contracting \( E \) onto a \( \mathbb{P}^1 \). A computation shows that \( X \) is Gorenstein, Fano, and log canonical along the image of \( E \). This fits into part (ii) of the above result because \( X' \), the dlt model (and resolution) of \( X \) was defined as a \( \mathbb{P}^2 \) bundle over \( C \).

Lastly, we can immediately generalize this result to the case of non-normal slc varieties with anti-ample canonical sheaf and a strictly log canonical singularity. This is equivalent to studying the case of a pair \((X, \Delta)\) where \(-K_X + \Delta\) is ample and the 1-dimensional locus of log canonical singularities intersects \( \Delta \). Because the locus of log canonical singularities must intersect \( \Delta \) [kol92] but cannot be contained in \( \Delta \), \( X \) must in fact have a log canonical singularity along a curve. Then, a terminal modification \( X' \) of \( X \) and minimal model program on \( X' \) gives the same conclusion, where \( \Delta \) is considered as a component of \( E \).

In Section 5, we use these results to further analyze the moduli space of H-\( \epsilon \) stable pairs presented above.

Although the focus thus far has been on strictly log canonical varieties \( X \) with anti-ample canonical class, we also study the purely log terminal varieties. Explicitly understanding these varieties would be useful in classifying the singular varieties on the boundary of the moduli space of H-\( \epsilon \) stable pairs. In that vein, we will first focus on the canonical threefolds appearing in the moduli problem.

4.2. Canonical Fano threefolds. Much is known about canonical threefolds in general, and a standard reference is [Rei87]. In the Fano case, particularly when \( X \) is Gorenstein, such threefolds can be classified by invariants like \( K^3 X \) and the Fano index.

If \( X \) has at worst canonical singularities, the Fletcher-Reid plurigenera formula [Rei87, Theorem 10.2] gives the plurigenera of \( X \) in terms of \( K^3 X \), \( \chi(O_X) \), and coefficients \( c_P \) determined by a basket of singularities for \( X \). In the proof of the theorem, Reid shows that the coefficients \( c_P \) can be computed in terms of the finitely many points \( Q_i \) such that \( K_{X'} \) is not Cartier at \( Q_i \), where \( X' \to X \) is a crepant partial resolution such that \( X' \) has only terminal singularities.

In [Fle89, Theorem 1.1], Fletcher shows the plurigenera formula is exact, meaning that any two canonical threefolds with the same plurigenera have the same \( K^3 X \), \( \chi(O_X) \), and basket of singularities. The contribution from the singularities is nonzero precisely when there are points \( Q_i \) such that \( K_{X'} \) is not Cartier at \( Q_i \). In our case, because \( X \) is a flat degeneration of \( \mathbb{P}^3 \), the plurigenera
of $X$ and $\mathbb{P}^3$ are the same, so this inversion of the plurigenus formula implies that $X'$ must be a terminal Gorenstein variety. Because $X' \to X$ is any crepant partial resolution such that $X'$ has only terminal singularities, $K_{X'}^3 = -64$ and we can take $X'$ to be $\mathbb{Q}$-factorial.

Although there are potentially many canonical degenerations of $\mathbb{P}^3$, there are not many terminal degenerations. Namely, there is only $\mathbb{P}^3$.

**Theorem 4.19.** If $X$ is a terminal variety that admits a smoothing to $\mathbb{P}^3$, then $X \cong \mathbb{P}^3$.

**Proof.** The Fletcher-Reid plurigenus formula shows that if $X$ is not Gorenstein, it does not admit a smoothing to $\mathbb{P}^3$, so it suffices to consider Gorenstein threefolds $X$. In this case, [CJR08, Theorem 2.1] implies that the Fano index of $X$, the maximal integer $r$ such that $K_X \sim -rH$ for $O(H) \in \text{Pic}(X)$, is equal to that of $\mathbb{P}^3$. Therefore, the Fano index of $X$ is 4. Then, [CJR08, Theorem 3.1] says that, because the Fano index is maximal, $X \cong \mathbb{P}^3$. \hfill $\Box$

There do exist non-trivial canonical degenerations of $\mathbb{P}^3$. The following is an example of such a variety, pointed out by Hacking.

**Example 4.20.** First, observe that the standard embedding of the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is an element of the linear system $\mathcal{O}_{\mathbb{P}^3}(2)$. Then, let $Z$ be the image of the Veronese embedding of $\mathbb{P}^3 \rightarrow \mathbb{P}^9$. There is a standard degeneration from $Z$ to the cone over a hyperplane section of $Z$ by taking the cone over $Z$ (see, for example, [KM98, Example 7.61]). In this case, the hyperplane section of $Z$ corresponds to an element of $\mathcal{O}_{\mathbb{P}^3}(2)$, and is the $O(2,2)$ embedding of the quadric surface in $\mathbb{P}^5$. A computation shows that the cone over this is indeed Gorenstein as it is the cone over the anti-canonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1$. A check shows that this has canonical singularities; for details see [Kol13, Lemma 3.1]. Therefore, this gives an example of a flat degeneration of $\mathbb{P}^3$ to a Gorenstein, strictly canonical variety.

Although there is no restriction on the degree of $d$, in the moduli problem at hand, if $D$ misses the singular point of $X$, consider the strict transform $D'$ in the resolution $\pi : X' \to X$ obtained by blowing up the singular point. The singularity is canonical and $\pi^*K_X = K_{X'}$. However, $X'$ is the projectivization of a vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ and admits a morphism $X' \to \mathbb{P}^1 \times \mathbb{P}^1$ contracting the fibers. Because each fiber $F \cong \mathbb{P}^1$, $K_{X'} \cdot F = -2$. This implies $K_X \cdot \pi(F) = -2$. Because $D$ misses the singular point of $X$, $D \cdot \pi(F) \in \mathbb{Z}$, so the relationship $dK_X + 4D \sim 0$ implies $D \cdot \pi(F) = -2dK_X \cdot \pi(F) = -\frac{d}{2}$. Therefore, $d$ must be even.

If $d$ is odd, this implies that $D$ must contain the singular point of $X$. For examples when $d = 5$, see Section 6.

A priori there are many canonical degenerations of $\mathbb{P}^3$, but the following theorem shows that if $d$ is odd, they must be closely related to the previous example. In fact, they must be $\mathbb{P}^3$ or cones over the Veronese of elements of the linear system $|\mathcal{O}_{\mathbb{P}^3}(2)|$, which have a simple description.

**Theorem 4.21.** For odd degree $d$, if $X$ is a canonical threefold appearing in an $H$-stable pair $(X,D)$ of degree $d$, then $X$ is either $\mathbb{P}^3$, the cone over the Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, or the cone over the Veronese embedding of the quadric cone, also known as $\mathcal{P}(1,1,2,4)$.

If $X$ has only terminal singularities, Theorem 4.19 implies the result. If $X$ has canonical singularities, consider a crepant partial resolution $X' \to X$ such that $X'$ is terminal and $\mathbb{Q}$-factorial. Before giving the proof, we give a sketch of the argument.

By [Fle89], if $K_{X'}$ is not Cartier, there is a nonzero contribution to a basket of singularities on $X$, so $X$ is not isomorphic to $\mathbb{P}^3$. It then suffices to consider the case where $X'$ is a terminal, $\mathbb{Q}$-factorial Gorenstein variety with $-K_{X'}$ nef.

Running a minimal model program on $X'$, if it terminates in a morphism $X' \dashrightarrow Y \to \text{Spec} \ k$, then $Y$ must be a terminal Fano threefold with $\rho(Y) = 1$. Studying the pseudo-index of $Y$ as in [CJR08] and combining this with the fact that $K_{X'}^3 = -64$ would imply that $X'$ itself must have been $\mathbb{P}^3$, so $X \cong \mathbb{P}^3$. If a run of the minimal model program on $X'$ terminates in a morphism
$X' \to Y \to C$, where $C$ is a curve, the generic fiber of $Y \to C$ must be a smooth del Pezzo surface, so there are sufficiently general curves $L \subset X'$ such that $K_{X'} \cdot L = -3$ or $-2$, and if the termination is in a surface $W$, there are sufficiently general curves $L \subset X'$ such that $K_{X'} \cdot L = -2$. If any of these curves miss the exceptional divisors of the partial resolution $\pi : X' \to X$, then $D \cdot \pi(L) \in \mathbb{Z}$, and we can argue as in the example above to show that $d$ must be even. Similarly, we can reach the same conclusion if $D$ does not pass through the strictly canonical singularities of $X$.

The remaining case is when $D$ contains the strictly canonical singularities of $X$ and the general fiber $L$ intersects the exceptional divisors of $\pi : X' \to X$, because it is not obvious that $D \cdot \pi(L) \in \mathbb{Z}$. However, if this occurs, one can exactly understand the fibration.

First, let us recall results of Cutkosky on contractions of extremal rays on terminal, $\mathbb{Q}$-factorial Gorenstein threefolds.

**Lemma 4.22.** [Cut88, Lemma 2] Suppose that $X$ is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold. Then, $X$ is factorial.

**Lemma 4.23.** [Cut88, Lemma 3] Suppose that $X$ is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold and $\phi : X \to Y$ is the contraction of a $K_X$-negative extremal ray with at most one dimensional fibers. Then, $Y$ is factorial. In particular, $Y$ is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold, and $\phi$ cannot be a small contraction.

**Theorem 4.24.** [Cut88, Theorem 4] Suppose that $X$ is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold and $\phi : X \to Y$ is a birational contraction of a surface $W \subset X$ to a curve $C \subset Y$. Then, $Y$ is smooth near $C$.

**Theorem 4.25.** [Cut88, Theorem 5] Suppose that $X$ is a terminal, $\mathbb{Q}$-factorial Gorenstein threefold and $\phi : X \to Y$ is a birational contraction of a surface $W \subset X$ to a point $p \subset Y$. Then, one of the four cases below occur:

(i) $Y$ is nonsingular near $p$, $W \cong \mathbb{P}^2$, and $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$.
(ii) $W \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$.
(iii) $W$ is isomorphic to a reduced, irreducible singular quadric surface $D$ in $\mathbb{P}^3$ and $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_D$.
(iv) $Y$ is singular at $p$, $W \cong \mathbb{P}^2$, and $\mathcal{O}_W(W) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$.

Now we can prove Theorem 4.21.

**Proof.** Let us begin with the simplest case: no component of the locus of canonical singularities is contained in $D$. Then, the contraction of a $K_{X'}$-negative extremal ray must be birational $X' \to Y$ or a Fano fibration $X' \to S$ or $X' \to C$, where $\dim S = 2$, $\dim C = 1$:

$$
\begin{array}{c}
X' \\
\downarrow \\
Y \\
\downarrow \\
S \\
\downarrow \\
C 
\end{array}
$$

Because $-K_{X'}$ is nef and non-trivial, the contraction cannot be $X' \to \text{Spec} \, k$. Also, by Lemma 4.23, $X' \to Y$ is necessarily a divisorial contraction. Therefore, in every case, the generic curve contracted has $K_{X'} \cdot C = -1, -2$, or $-3$, so the image of $C$ on $X$ has $K_X \cdot \pi(C) = -1, -2$, or $-3$. Because $D$ does not contain the locus of canonical singularities, for a sufficiently generic curve $C$, $D \cdot \pi(C) \in \mathbb{Z}$. Therefore, the relationship $dK_X + 4D \sim 0$ implies $d$ is even.

If a component $\Delta$ of the locus of canonical singularities is contained in $D$, we can separate into two cases: either $\Delta$ is one- or zero-dimensional.

**Case 1.** $\dim \Delta = 1$. 


If $\Delta$ is one-dimensional, consider the partial resolution $\pi : X' \to X$. Because $X$ has only canonical singularities, the fibers of $\pi$ must be chains of rational curves. We can study the pullback $\pi^*D$: in particular, $\pi^*D = \tilde{D} + \sum a_i F_i$, where $\tilde{D}$ is the strict transform of $D$ and $F = \bigcup F_i$ is the fiber over $\Delta$. Let $F_0$ be a component of $F$ such that $\dim \pi(F_0) = 1$. For a generic curve $C \subset F_0$ contracted by $\pi$, $K_{X'} \cdot C = 0$ and $F \cdot C < 0$. However, $-2 = K_{F_0} \cdot C = (K_{X'} + F_0) \cdot C$, so there can be at most one component $F_i$ meeting $C$ with $F_i \cdot C = 1$. Therefore, either there is no such $F_i$ and

$$0 = \pi^* D \cdot C = \tilde{D} \cdot C + \sum a_i F_i \cdot C = n + a_0(-2)$$

so $a_0 \in \mathbb{Z}[1/2]$ or there is some $F_i$ that meets $C$ and a contracted curve $C' \subset F_i$ meeting $F_0$ such that

$$0 = \pi^* D \cdot C = \tilde{D} \cdot C + \sum a_i F_i \cdot C = n + a_0(-2) + a_i(1)$$

$$0 = \pi^* D \cdot C' = \tilde{D} \cdot C' + \sum a_i F_i \cdot C' = m + a_0(1) + a_i(-2)$$

so $a_0, a_i \in \mathbb{Z}[1/3]$.

This shows for generic curves in $X$ meeting $D$, the intersection with $D$ is in $\mathbb{Z}[1/3]$. With this in mind, now contract a $K_{X'}$ negative extremal ray on $X'$. As above, we have the following options:

![Diagram](image)

**Case A.**

Assume first $X' \to Y$ is divisorial.

If $X' \to Y$ is divisorial and with at most one dimensional fibers, the generic fiber $C$ has $K_{X'} \cdot C = -1$, so the image in $X$ has $K_X \cdot \pi(C) = -1$. For sufficiently generic $C$, $D \cdot \pi(C) \in \mathbb{Z}$. Therefore, the relationship $dK_X + 4D \sim 0$ implies $d$ must be even. If $X' \to Y$ is divisorial but contracts a surface to a point, if any case other than (i) occurs as in Theorem 4.25, we still find a generic curve $C$ in the fiber with $K_{X'} \cdot C = -1$.

If case (i) occurs, the threefold $Y$ is still terminal, $\mathbb{Q}$-factorial, and Gorenstein, so we can contract a new $K_{Y'}$ negative extremal ray and repeat. If at any point our contraction one of the cases (ii), (iii), or (iv), by the same argument above, we are done. If we perform a divisorial contraction with at most one-dimensional fibers, again the output is terminal, $\mathbb{Q}$-factorial and Gorenstein, so we can continue. Therefore, it suffices to analyze the possible fibrations that arise as minimal models of a terminal, $\mathbb{Q}$-factorial, Gorenstein variety $X'$ where, at each step of the minimal model program, the resulting variety is also terminal, $\mathbb{Q}$-factorial, and Gorenstein.

However, after some number of divisorial contractions, we reach the point of a fibration, then the divisorial contractions were blow ups of some point(s) on the fibration. Therefore, either the general fiber of the fibration doesn’t intersect $F$, or after blowing up, a fiber of the divisorial contraction doesn’t intersect $F$. Therefore, its image on $X$ has $D \cdot C \in \mathbb{Z}$. Arguing as above implies $d$ is even.

Therefore, the only two cases that remain to be studied are if the only possible $K_{X'}$ negative contraction yields a fibration.

**Case B.**

If $\phi : X' \to S$ is a fibration with general fiber $\cong \mathbb{P}^1$, either there are $F$-trivial fibers $C$ or $F$ is relatively ample. In the first case, $K_X \cdot \pi(C) = -2$ and $D \cdot \pi(C) \in \mathbb{Z}$, so $d$ be even. Assume then that $F$ is relatively ample. By [Cut88, Theorem 7], $S$ must be smooth and $X'$ must be a conic bundle over $S$. If $X' \to S$ has any singular fibers, then there exist curves $C$ such that $K_{X'} \cdot C = -1$,
and we argue as before to show \( d \) must be even. Therefore, we may assume every fiber is smooth and \( X' \to S \) is a smooth \( \mathbb{P}^1 \)-bundle over a smooth surface \( S \). Furthermore, by [CJR08, Lemma 2.5], \( -K_S \) is big and nef. Because \( F \) is relatively ample, for some component \( F_0 \) of \( F \), the induced morphism \( F_0 \to S \) must be finite.

However, \( F_0 \) is contractible on \( X' \), so we have a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow{\phi} & & \\
S & & \\
\end{array}
\]

Consider a smooth curve \( C \subset S \) such that \( Z = \phi^{-1}(C) \) contains a contracted curve in \( F_0 \). Because every fiber of \( \phi \) is \( \mathbb{P}^1 \), \( Z \) is a ruled surface over \( C \) and because \( F_0 \to S \) is finite, \( F_0|_Z \) is a multisection of \( \phi|_Z : Z \to C \). However, this multisection is contractible in \( Z \) to a surface \( Z \subset X \). For generic \( Z \), \( Z \) is not contracted by \( \pi \), so intersection theory on ruled surfaces implies that \( F_0|_Z \) is actually a section.

This is true for any such \( Z \), so the degree of \( \phi|_{F_0} : F_0 \to S \) must be 1, hence \( S \cong F_0 \) and \( F_0 \) is a section of \( \phi \).

Assume first that \( F_0 \) is contracted to a curve via \( \pi : X' \to X \). Then, \( S \cong F_0 \) must be a ruled surface over \( C \) with \( -K_S \) big and nef, so \( S \) must be \( \mathbb{P}^1 \times \mathbb{P}^1, F_1, \) or \( \mathbb{F}_2 \).

Because each of these surfaces have \( \rho(S) = 2 \), it follows that \( \rho(X') = 3 \) and there are at most two components of \( F \). If there is only one component of \( F \), there is exactly one contraction \( X' \to X \) that is \( K_{X'} \)-trivial, but \( \rho(X') \geq 3 \) implies that there are at least two \( K_{X'} \)-negative contractions. One corresponds to the map \( \phi : X' \to S \cong F \) and the other must correspond to another case, so we use the argument in the other cases to find a contradiction or show \( X \cong \mathbb{P}(1, 1, 2, 4) \).

If there are two components of \( F \), so \( F = F_0 \cup F_1 \), then \( F_1 \) must also be relatively ample as it is covered by \( K_{X'} \)-trivial curves so cannot be contracted by \( \phi \). Therefore, \( S \cong F_0 \cong F_1 \). However, this is only possible if both \( F_0 \) and \( F_1 \) are contracted to a curve via \( \pi \); otherwise, say \( F_1 \) is contracted to a point, then there exist \( F_1 \) trivial curves intersecting \( F_0 \), so \( F_1 \) is not relatively ample.

Now, as above, consider a smooth curve \( C \subset S \) such that \( Z = \phi^{-1}(C) \) contains a contracted curve in \( F_0 \). Because every fiber of \( \phi \) is \( \mathbb{P}^1 \), by the argument above, \( Z \) is a ruled surface over \( C \) with a contractible section. However, \( Z \) must also contain a contracted curve in \( F_1 \), hence \( Z \) has two contractible sections. However, this is a contradiction, as it would imply \( \pi|_Z : Z \to \pi(Z) \) contracts \( Z \) to a curve.

Therefore, we can assume that \( F_0 \) is contracted to a point via \( \pi : X' \to X \). If each \( \phi \)-ample divisor \( F_i \) does not intersect \( D \), we find curves \( C \) such that \( D \cdot C \in \mathbb{Z} \) and \( K_X \cdot C = -2 \) so \( d \) is even. Therefore, it suffices to consider only \( F_i \) that intersect \( D \). Because \( D \) contains \( \Delta \), there is some \( F_1 \) that is contracted to a curve via \( \pi \) such that \( F_1 \cap D \neq \emptyset \) and \( F_1 \cap F_0 \neq \emptyset \). The intersection \( F_1 \cap F_0 \) must be a fiber of the ruled surface \( F_1 \), hence \( F_0 \) contains a curve \( C \) such that \( K_{X'} \cdot C = 0 \), \( F_1 \cdot C = -2 \), and \( F_0 \cdot C = 0 \). Therefore, \( K_{F_0} \cdot C = 0 \) so \( F_0 \cong \mathbb{F}_2 \).

This implies that \( \rho(X') = 3 \) because \( \phi : X' \to S \cong \mathbb{F}_2 \) is a \( \mathbb{P}^1 \) bundle, hence there is only one exceptional divisor \( F_1 \) with \( \dim \pi(F_1) = 1 \). If \( F_1 \) were also \( \phi \)-ample, we must have \( F_1 \cong \mathbb{F}_2 \) also be a section. However, the intersection curve \( C = F_0 \cap F_1 \) is a section of \( F_0 \) but a fiber of \( F_1 \), a contradiction.

Therefore, \( F_1 \) is not \( \phi \)-ample, so we must have \( F_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \). Contracting \( F_0 \) and \( F_1 \) to \( X \) shows \( \rho(X) = 1 \) and \( X \) has a \( \frac{1}{2}(1, 1, 2) \) singularity, hence we must have \( X \cong \mathbb{P}(1, 1, 2, 4) \).

**Case C.**

If \( \phi : X' \to C \) is a fibration with general fiber a smooth del Pezzo surface and \( C \cong \mathbb{P}^1 \), we can first note that if the general fiber is a surface other than \( \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \), there exist curves \( C \) with
$K_{X'} \cdot C = -1$, so we argue as before to conclude $d$ is even. Similarly, if the fiber is $\mathbb{P}^2$, there exist curves $C$ with $K_{X'} \cdot C = -3$, and again we can conclude $d$ is even. Therefore, it suffices to analyze the case when the general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$. Because $\rho(C) = 1$ and $\phi$ was an extremal contraction, $\rho(X') = 2$. There is then only one component of $F$. We would like to show that there are divisors $D_1$ and $D_2$ whose restriction to each fiber are the different rulings. Those are not linearly equivalent nor are they linearly equivalent to the general fiber $F$, hence it would imply $\rho(X') \geq 3$, a contradiction.

Suppose for contradiction $X'$ does exist. If $F$ was contained in a fiber of $\phi$, then there exist many $F$-trivial curves with $K_{X'} \cdot C = -2$, and on $X$, $D \cdot \pi(C) \in \mathbb{Z}$. As usual, we consider the relation $dK_X + 4D \sim 0$, so find that $d$ must be even.

Now consider the case that $F$ is $\phi$-ample, so $\phi|_F : F \to C$ gives $F$ the structure of a ruled surface over $C$ and contracts only $K_{X'}$-negative curves. Because $\pi|_F$ also contracts $F$ to a curve but contracts only $K_{X'}$-trivial curves, $F$ must have the structure of a product, so $F \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\Gamma \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a fiber of $\phi$. We claim that the span of $F$, $\Gamma$, and $K_{X'}$ in $N^1(X)$, so we must have $\rho(X') \geq 3$, a contradiction. To see the claim, note that $\Gamma|_F$ must be a ruling of $F$, so $\Gamma|_F \in |\mathcal{O}_F(1,0)|$. Next, observe that $K_{X'}|_F$ is negative on the fibers contracted by $\phi$ and trivial on the fibers contracted by $\pi$. However, these are the two rulings of $F$, so $K_{X'}|_F \in |\mathcal{O}_F(-2,0)|$. Furthermore, $K_{X'}$ and $\Gamma$ are certainly not linearly equivalent. Finally, consider $F|_F$. On the fibers of $F$ contracted by $\pi$, by the negativity lemma, this must be negative, so $F|_F \in |\mathcal{O}_F(a,-b)|$ for $b > 0$. Therefore, $F$ cannot be linearly equivalent to any linear combination of $\Gamma$ and $K_{X'}$, so $\rho(X') \geq 3$, so $X'$ cannot exist.

**Case 2.** $\dim \Delta = 0$.

Lastly, suppose the locus of log canonical singularities is a point contained in $D$. We can study the same contractions:

\[ \begin{array}{ccc}
  X' & \rightarrow & Y \\
  \downarrow & & \downarrow \\
  C & \rightarrow & S \\
  \downarrow & & \downarrow \\
  A & \rightarrow & B \\
  \end{array} \]

In this case, if $F$ is the exceptional locus of the map $\pi : X' \to X$, the curves in $F$ are all $K_{X'}$-trivial, so cannot be contracted by a $K_{X'}$-negative contraction. Therefore, the third arrow (Case C) $X' \to C$ is not possible.

**Case A.**

Because the curves in $F$ are all $K_{X'}$-trivial, the only possible $K$-negative divisorial contraction over $F$ is $X' \to Y$ that has at most one dimensional fibers. However, then $Y$ would be terminal, Gorenstein, and $Q$-factorial, so we can continue the minimal model program on $Y$. Much of this argument is the same as Case A above. If divisorial contractions happen first, there will exist curves with $K_{X'} \cdot C$ equal to $-1$, $-2$, or $-3$ that don’t intersect $F$, and (invoking factoriality of $X'$), $D \cdot \pi(C) \in \mathbb{Z}$, and we can conclude $d$ is even.

**Case B.**

The remaining case is if the only $K_{X'}$-negative contraction is a fibration $X' \to S$, and as in Case B above, we can assume every fiber is smooth and isomorphic to $\mathbb{P}^1$.

Therefore, we find ourselves in the situation where $\phi : X' \to S$ is a smooth $\mathbb{P}^1$-bundle over a smooth surface $S$ and by [CJR08, Lemma 2.5], $-K_S$ is big and nef. Exactly as above, we can conclude $S \cong F_0$ for some component $F_0$ of $F$ and $F_0$ is a section of $\phi$.

Furthermore, for any component $F_i$ of $F$, because $\pi(F_i)$ is a point, $K_{X'} \cdot C_i = 0$ for any $C_i \subset F_i$, so $C_i$ cannot be contracted by $\phi$. Therefore, every component $F_i$ of $F$ is $\phi$-ample and $S \cong F_i$ for all $i$.

Briefly turning our attention to the map $\pi : X' \to X$, because components of $F$ are contracted to points by $\pi$, every curve in $F$ is $K_{X'}$-trivial and $F$-negative. By adjunction, for $C \in F_i$, $(K_{X'} + F_i) \cdot
\[ C = K_F \cdot C, \text{ so } F_i \cdot C = K_{F_i} \cdot C. \] Therefore, not only is \(-K_S = -K_{F_i}\) big and nef, but it is ample, so \(S \cong F_i\) is a Fano surface. If there are any \(-1\) curves on \(F\), taking the intersection product with \(\pi^* D = D + aF\) implies \(a \in \mathbb{Z}\), so for any curve \(C\) on \(X\), \(D \cdot C \in \mathbb{Z}\). Therefore, for a fiber of \(\phi\) with \(K_X \cdot C = -2\), we find that \(d\) must be even. Similarly, if \(F_i \cong \mathbb{P}^2\), we find lines with \(F \cdot C = -3\), so \(D \cdot C \in \mathbb{Z}[1/3]\) and the same conclusion holds.

Therefore, the only remaining case is if \(F_i \cong \mathbb{P}^1 \times \mathbb{P}^1\) for all \(i\):

\[
\begin{array}{c}
X' \xrightarrow{\pi} X \\
\phi \downarrow \\
\mathbb{P}^1 \times \mathbb{P}^1
\end{array}
\]

and \(\pi : X' \to X\) contracts \(F\).

Consider \(Z = \phi^{-1}(C)\) for a generic ruling \(C\) on \(\mathbb{P}^1 \times \mathbb{P}^1\). By construction, each \(F_i|_Z\) is a contractible section of the smooth ruled surface \(Z\). Because \(Z\) is not contracted by \(\pi\), this implies that there is only one \(F_i\) and \(F = F_0\).

Then, \(X\) is locally isomorphic to the cone over the anticanonically embedded \(\mathbb{P}^1 \times \mathbb{P}^1\), Example 4.20. However, \(\rho(X) = 1\), so \(X\) must actually be isomorphic to that cone.

\[ \square \]

Ultimately, the odd degree pairs are behaving in a very special way: oddness of the degree is forcing constraints on the threefolds \(X\) that can appear. Summarizing the previous two sections, no log canonical threefolds \(X\) can appear and the only canonical threefold allowed is \(\mathbb{P}^3\). That leads us to the following section.

### 4.3. Log terminal degenerations of \(\mathbb{P}^3\).

To completely classify the ambient space in the odd degree case, it remains to understand log terminal threefolds \(X\) that are degenerations of \(\mathbb{P}^3\). We will approach this in general (not only in the odd case). One should note that, although we’ve focused our attention only on the ambient threefolds \(X\), it is ‘enough’ to classify only these threefolds: if we know \(X\), we can determine all possible \(D\) by varying \(D \in |-\frac{4}{3}K_X|\).

There are natural log terminal varieties to consider: weighted projective space. We summarize the case in dimension 2, due to Manetti and Hacking.

**Theorem 4.26** (Manetti). If \(X\) is a normal, log terminal degeneration of \(\mathbb{P}^2\) such that the total space is \(\mathbb{Q}\)-Gorenstein, then \(X \cong \mathbb{P}(p^2, q^2, r^2)\) or a smoothing of such a space, where

\[ 3pqr = p^2 + q^2 + r^2. \]

Furthermore, all such varieties admit a \(\mathbb{Q}\)-Gorenstein smoothing to \(\mathbb{P}^2\).

In addition to the theorem, we can describe all solutions with an infinite graph:

**Theorem 4.27** (Hacking). All solutions to

\[ 3pqr = p^2 + q^2 + r^2 \]

can be obtained by starting with the obvious solution \((1, 1, 1)\) and performing a sequence of mutations: if \((p, q, r)\) is a solution, then \((p, q, 3pq - r)\) is a solution.

One could hope for such an analog in the three dimensional case, although that seems out of reach: the proof of this theorem heavily relies on the classification of (all) surface singularities. However, there are partial results, using properties of weighted projective spaces. For background on weighted projective space, we refer the reader to [Dol82] or [IF00].

First, recall that a weighted projective space \(\mathbb{P}(a_0, \ldots, a_n)\) is called well-formed if every subset of \(n\) of the \(a_i\) has no common factors. For example, \(\mathbb{P}(1, 2, 4)\) is not well-formed, but is isomorphic
to $\mathbb{P}(1,1,2)$, which is. We will call the set of integers $(a_0, \ldots, a_n)$ well-formed if the associated weighted projective space is.

**Proposition 4.28.** If $\mathbb{P}(a, b, c, d)$ is well-formed and admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^3$, then

$$64abcd = (a + b + c + d)^3.$$  

**Proof.** In order for $X = \mathbb{P}(a, b, c, d)$ to have a $\mathbb{Q}$-Gorenstein smoothing, $K_X^3 = K_{\mathbb{P}^3}^3 = -64$. But, $\mathcal{O}(K_X) = \mathcal{O}(-a - b - c - d)$, and

$$K_X^3 = \frac{(-a - b - c - d)^3}{abcd}.$$  

Therefore, we must have

$$64abcd = (a + b + c + d)^3.$$  

\hfill \Box

**Remark 4.29.** This formula is distinctly different from the two dimensional version; indeed the previous version has been simplified from this form. Analyzing the square of the canonical divisor would say $X = \mathbb{P}(a, b, c)$ could only smooth to $\mathbb{P}^2$ if

$$9abc = (a + b + c)^2.$$  

Taking square roots of both sides and applying a little bit of number theory implies that $a$, $b$, and $c$ have to be perfect squares. Setting $a = p^2$, $b = q^2$, $c = r^2$ gives the above version.

One could make the immediate generalization to $n$-dimensional weighted projective spaces:

**Proposition 4.30.** If $\mathbb{P}(a_0, a_1, \ldots, a_n)$ is well-formed and admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^n$, then

$$(n + 1)^n \Pi a_i = (\sum a_i)^n.$$  

The proof is the same as above.

Given the equation, one immediately questions if there are infinitely many solutions, or if there is a procedure for obtaining solutions as in the two dimensional case. There is certainly an infinite family of solutions, which makes sense geometrically. If we have a degeneration of $\mathbb{P}^2$ to such a weighted projective space, it should induce a degeneration of $\mathbb{P}^3$ to some sort of cone over that weighted projective space. This is the content of the following proposition, stated first in the three-dimensional case.

**Proposition 4.31.** If $\mathbb{P}(a, b, c)$ admits a smoothing to $\mathbb{P}^2$ (so $a = p^2$, $b = q^2$, $c = r^2$ in the previous theorem), then $d = \sqrt{abc} = \frac{a+b+c}{3} \in \mathbb{Z}$ and $P(a, b, c, d)$ satisfies the condition

$$64abcd = (a + b + c + d)^3.$$  

**Example 4.32.** Consider the surface $\mathbb{P}(1,1,4)$. This admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$ in the following way: consider the Veronese embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ and let $V \cong \mathbb{P}^2$ be the image. Taking the cone $C(V)$ over $V$, the general hyperplane section of $C(V)$ is isomorphic to $V$ and a special hyperplane section through the origin is isomorphic to the cone over a hyperplane section of $V$, or $\mathbb{P}(1,1,4)$. In other words, $\mathbb{P}^2$ and $\mathbb{P}(1,1,4)$ are hyperplane sections of the cone $\mathbb{P}(1,1,1,2)$. The criterion above says that $\mathbb{P}(1,1,4,2) \cong \mathbb{P}(1,1,2,4)$ satisfies the necessary condition to admit a smoothing to $\mathbb{P}^3$. Indeed, such a smoothing exists. Consider the Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^{10}$ and the cone $\mathbb{P}(1,1,1,1,2)$ over the image of $\mathbb{P}^3$. The general hyperplane section is isomorphic to $\mathbb{P}^3$. A special hyperplane section through the origin is isomorphic to the cone over a hyperplane section of $\mathbb{P}^3 \subset \mathbb{P}^{10}$. However, given the cone $\mathbb{P}(1,1,2) \subset \mathbb{P}^3$, its image under the Veronese embedding is such a hyperplane section. In this embedding, the cone over $\mathbb{P}(1,1,2)$ is $\mathbb{P}(1,1,2,4)$, as desired.
Proof. Because $a, b, c$ are perfect squares, we have $d \in \mathbb{Z}$. But, because $\mathbb{P}(a, b, c)$ admits a smoothing to $\mathbb{P}^2$,
\[ 3d = a + b + c \]
so
\[ (a + b + c + d)^3 = (4d)^3 = 64d^3 = 64abcd. \]
\[ \square \]

Proposition 4.33. If $\mathbb{P}(a_0, a_1, \ldots, a_n)$ satisfies
\[ (n + 1)^n \Pi a_i = (\sum a_i)^n, \]
then $b = (\Pi a_i)^{1/n} = \sum a_i^{n+1}/(n+1) \in \mathbb{Z}$ and $\mathbb{P}(a_0, a_1, \ldots, a_n, b)$ satisfies
\[ (n + 2)^{n+1} b \Pi a_i = (b + \sum a_i)^{n+1}. \]

The proof is the same as above.

These propositions only imply that these weighted projective spaces satisfy the necessary conditions to admit a smoothing to $\mathbb{P}^n$, not that it is sufficient.

At least in the three dimensional case, to understand if these weighted projective spaces could smooth to $\mathbb{P}^3$, one would have to study the versal deformation theory of these cyclic quotient singularities. It is enough to work locally: by standard cohomology calculations for weighted projective space, local to global deformations are unobstructed, as $H^2(X, T_X) = 0$.

Let us further investigate the three-dimensional case. Using a computer, one can list the integer solutions to the equation $64abcd = (a + b + c + d)^3$ such that the associated weighted projective space is well-formed, and finds the following weighted projective spaces with $a, b, c, d \leq 125$:

\[
\begin{align*}
\mathbb{P}(1, 1, 1, 1) \\
\mathbb{P}(1, 1, 2, 4) \\
\mathbb{P}(1, 2, 9, 12) \\
\mathbb{P}(1, 4, 10, 25) \\
\mathbb{P}(1, 4, 16, 27) \\
\mathbb{P}(1, 6, 9, 32) \\
\mathbb{P}(1, 7, 27, 49) \\
\mathbb{P}(1, 9, 50, 60) \\
\mathbb{P}(1, 22, 32, 121) \\
\mathbb{P}(3, 4, 63, 98)
\end{align*}
\]

There are some hidden patterns in the data. First off, in light of Proposition 4.31, there are some solutions $\mathbb{P}(a, b, c, d)$ arising from the degenerations $\mathbb{P}(a, b, c)$ of $\mathbb{P}^2$ where $d = \frac{a+b+c}{3}$ is the average of $a$, $b$, and $c$. The only three that appear in this truncated list are $\mathbb{P}(1, 1, 1, 1)$, $\mathbb{P}(1, 1, 2, 4)$, and $\mathbb{P}(1, 4, 10, 25)$. These solutions are well understood and, following work of Hacking and Manetti (using work of Markov), we have the following result. This is simply a restatement of Theorem 4.27, adding in the fourth variable $d$.

Proposition 4.34. There is an infinite family of well-formed solutions to the equation $64abcd = (a + b + c + d)^3$ given by $(a, b, c, d) = (\alpha^2, \beta^2, \gamma^2, \alpha \beta \gamma) = (\alpha^2, \beta^2, \gamma^2, \frac{\alpha^2 + \beta^2 + \gamma^2}{3})$. All such $\alpha, \beta, \gamma$ lie on an infinite tree and are obtained by a mutation of the form $(\alpha, \beta, \gamma) \rightarrow (\alpha, \beta, 3\alpha \beta - \gamma)$ starting from $(1, 1, 1)$.

In the list above, we see that $\mathbb{P}(1, 1, 1, 1)$, $\mathbb{P}(1, 1, 2, 4)$, and $\mathbb{P}(1, 4, 10, 25)$ are all of this form.

Definition 4.35. We will call a solution of this form $\mathbb{P}^2$-type because it arises from a degeneration of $\mathbb{P}^2$. 

The deformation theory of these weighted projective spaces is in general quite complicated. For instance, although they can be embedded into $\mathbb{P}^N$ for $N$ sufficiently large, they are in general not complete intersections and have high codimension. However, we can relate all solutions on this infinite tree as deformations of a common smoothing.

**Proposition 4.36** (Hacking). The weighted projective spaces appearing as solutions of $\mathbb{P}^2$-type can be connected as a family of threefolds over a two-parameter base, and are each $\mathbb{Q}$-Gorenstein deformations of a common smoothing.

**Proof.** This is proved in [Hac12, Example 7.7]. We relate the weighted projective spaces one step apart on the infinite tree over a two-parameter base.

Let $\mathbb{P}(a, b, c, d)$ and $\mathbb{P}(a, b, c', d')$ be two solutions to $64abcd = (a + b + c + d)^3$ of $\mathbb{P}^2$-type related by one mutation so that

$$\mathbb{P}(a, b, c, d) = \mathbb{P}(\alpha^2, \beta^2, \gamma^2, \alpha\beta\gamma)$$

and

$$\mathbb{P}(a, b, c', d') = \mathbb{P}(\alpha^2, \beta^2, \gamma^2, \alpha\beta\gamma') = \mathbb{P}(\alpha^2, \beta^2, (3\alpha\beta - \gamma)^2, \alpha\beta(3\alpha\beta - \gamma)).$$

Using the fact that $3\alpha\beta\gamma = \alpha^2 + \beta^2 + \gamma^2$ (and similarly for $\gamma'$), we can form the two-parameter family

$$X : x_0 x_1 = s x_2^\gamma + t x_3^\gamma \subset \mathbb{P}(\alpha^2, \beta^2, \gamma, \alpha\beta) \times \mathbb{A}^2_{s,t},$$

of weighted degree $\alpha^2 + \beta^2 = \gamma\gamma'$ threefolds in $\mathbb{P}(\alpha^2, \beta^2, \gamma, \alpha\beta').$

When $s = t = 0$, we get a non-normal threefold $\mathbb{P}(\alpha^2, \gamma, \alpha\beta) \cup \mathbb{P}(\beta^2, \alpha\beta).$

When $s = 0$ but $t \neq 0$, we get $\mathbb{P}(\alpha^2, \beta^2, \gamma^2, \alpha\beta\gamma)$ via the degree $\gamma$ embedding

$$\mathbb{P}(\alpha^2, \beta^2, \gamma^2, \alpha\beta\gamma) \rightarrow (x_0 x_1 = t x_3^\gamma) \subset \mathbb{P}(\alpha^2, \beta^2, \gamma, \alpha\beta)$$

given by

$$(u, v, w, t) \mapsto (x_0, x_1, x_2, x_3, x_4) = (u\gamma, v\gamma, w, u v, t).$$

When $s \neq 0$ but $t = 0$, we get $\mathbb{P}(\alpha^2, \beta^2, \gamma^2, \alpha\beta\gamma')$ via the degree $\gamma'$ embedding

$$\mathbb{P}(\alpha^2, \beta^2, \gamma^2, \alpha\beta\gamma') \rightarrow (x_0 x_1 = s x_2^\gamma) \subset \mathbb{P}(\alpha^2, \beta^2, \gamma, \alpha\beta)$$

given by

$$(u, v, w, t) \mapsto (x_0, x_1, x_2, x_3, x_4) = (u\gamma', v\gamma', u v, w, t).$$

Finally, for $s \neq 0$ and $t \neq 0$, we get a smoothing of the singularities of index $c$ and $c'$, respectively. Because this is taking place as a complete intersection in weighted projective space, which is $\mathbb{Q}$-factorial, the total space of these smoothings is $\mathbb{Q}$-Gorenstein. □

There is a second infinite tree of solutions to this equation, almost none of which are of $\mathbb{P}^2$-type.

**Proposition 4.37.** There is an infinite family of well-formed solutions to the equation $64abcd = (a + b + c + d)^3$ given by $(a, b, c, d) = (a, b, c, a + b + c)$. All such $(a, b, c, d)$ lie on an infinite tree and are obtained by a mutation of the form $(a, b, c, d) \rightarrow (a, b, 8ab - a - b - d, 8ab - d)$ starting from $(1, 1, 2, 4)$.

**Proof.** We give a sketch of the proof, verifying some of the necessary details. If $a + b + c = d$, the equation $64abcd = (a + b + c + d)^3$ simplifies to $8abc = (a + b + c)^2$. If desired, one can simplify this further by showing $a = c^2$, $b = \beta^2$, and $c = \gamma^2$ so the equation becomes $4\alpha\beta\gamma = \alpha + \beta + 2\gamma$.

We can regard the equation $8abc = (a + b + c)^2$ or $4\alpha\beta\gamma = \alpha + \beta + 2\gamma$ as having two variables fixed, quadratic in the other, and replace one root with another to get the desired mutation.

For more details, we direct the interested reader to [KN98]. □

In the list above, one sees that $\mathbb{P}(1, 1, 2, 4)$, $\mathbb{P}(1, 2, 9, 12)$, and $\mathbb{P}(1, 9, 50, 60)$ are all of this form.

**Definition 4.38.** We will call a solution of this form sum-type because one entry is the sum of the others.
The proof of the proposition is similar to that of Proposition 4.34. One can prove a simple lemma showing that \((1,1,2,4)\) is the only overlap between the two families.

**Lemma 4.39.** The only solution to the equation \(64abcd = (a + b + c + d)^3\) that is both of \(\mathbb{P}^2\)-type and sum-type is \((1,1,2,4)\).

**Proof.** Assume \(a \leq b \leq d\) and \(a \leq c \leq d\). If \(\mathbb{P}(a,b,c,d)\) is of sum type, we must have \(d = a + b + c\) and without loss of generality, we can assume \(c = \frac{a + b + d}{3}\). Because the first solution is of sum-type, we must have \(8abc = (a + b + c)^2\) and, because the second is of \(\mathbb{P}^2\)-type, we must have \(9abd = (a + b + d)^2\).

The first equation is equivalent to \(8abc = 2d^2\) and the second to \(abd = c^2\), hence \(8c^3 = d^3\), so \(c = 2d\). From \(c = 2d\), we get \(a + b + c = 2c\), so \(a + b = c\), and \(2abc = c^2\) so \(2ab = c\). Therefore, \(a + b = 2ab\), hence \(a = b = 1\) and \(c = 2\) and \(d = 4\) \(\blacksquare\).

As in the case of solutions of \(\mathbb{P}^2\)-type, we can relate two weighted projective spaces of sum-type that are one mutation apart.

**Proposition 4.40.** Given two weighted projective spaces that are solutions of sum-type one mutation apart, there is a two-parameter \(\mathbb{Q}\)-Gorenstein family connecting them.

**Proof.** Let \((a,b,c,d)\) be the first solution and \((a,b,c',d') = (a,b,8ab - a - b - d,8ab - d)\) be the second. Without loss of generality, assume \(d < d'\). First, because \(d = a + b + c\), we have \(8abc = (a + b + c)^2 = d^2\). Then, observe that \(d(a + b) = d(d - c) = d^2 - dc = 8abc - dc = c(8ab - d) = cd'\), hence \(a + b = \frac{cd}{d}\), and similarly, \(a + b = \frac{cd'}{d'}\). Using this relationship repeatedly, we can form the desired family.

Then, we can consider the family

\[
\mathcal{X} : x_0x_1 = tx_2^{a+b} + sx_3^c \subset \mathbb{P}(ac,bc,c, a + b, d) \times \mathbb{A}^2_{s,t}.
\]

When \(s = 0\) and \(t = 0\), this is a non-normal threefold \(\mathbb{P}(ac,b, a + b, d) \cup \mathbb{P}(bc,c, a + b, d)\).

For \(t = 0\) but \(s \neq 0\), this is the image of the degree \(c\) embedding of

\[
\mathbb{P}(a,b,c,d) \rightarrow \mathbb{P}(a,b,c,a + b, d)
\]

given by

\[
(x, y, z, w) \mapsto (x_0, x_1, x_2, x_3, x_4) = (x^c, y^c, z, xy, w).
\]

When \(s = 0\) but \(t \neq 0\), this is the image of the degree \(a + b\) embedding of

\[
\mathbb{P}(a,b,c',d') \rightarrow \mathbb{P}(a,b,c,a + b, d)
\]

given by

\[
(x, y, z, w) \mapsto (x_0, x_1, x_2, x_3, x_4) = (x^{a+b}, y^{a+b}, xy, z, w).
\]

Finally, for \(s \neq 0\) and \(t \neq 0\), this gives a partial smoothing of the singularities of index \(c\) and \(d\) and \(c'\) and \(d'\).

Because the total space is a complete intersection in weighted projective space, it is \(\mathbb{Q}\)-Gorenstein. \(\blacksquare\)

**Remark 4.41.** Because \(\mathbb{P}^3\) is the ‘linear cone’ over the anticanonically embedded \(\mathbb{P}^2\), it makes sense that ‘cones’ (the weighted projective spaces \(\mathbb{P}(\alpha^2, \beta^2, \gamma^2, d)\)) over degenerations of \(\mathbb{P}^2\) are appearing as degenerations of \(\mathbb{P}^3\).

Analogously, the equation \(4\alpha\beta\gamma = \alpha + \beta + 2\gamma\) that appears in the course of studying solutions of sum-type parameterizes weighted projective spaces \(\mathbb{P}(\alpha^2, \beta^2, 2\gamma^2)\) that appear as degenerations of \(\mathbb{P}^1 \times \mathbb{P}^1\) [HP10, Theorem 1.2]. Because \(\mathbb{P}^3\) is a smoothing of the cone over the anticanonical embedding of \(\mathbb{P}^1 \times \mathbb{P}^1\), it makes sense that ‘cones’ (the weighted projective spaces \(\mathbb{P}(\alpha^2, \beta^2, 2\gamma^2, d)\)) over degenerations of \(\mathbb{P}^1 \times \mathbb{P}^1\) should be appearing as degenerations of \(\mathbb{P}^3\).
**Remark 4.42.** The fact that there are two essentially distinct families of solutions to the equation $64abcd = (a + b + c + d)^3$ already indicates the increase in complexity when studying degenerations of $\mathbb{P}^3$ versus those of $\mathbb{P}^2$. Furthermore, looking at the short list of given solutions above, one can observe that there are many in the list that do not appear in either of these two families. We are currently investigating other infinite families of solutions.

**Remark 4.43.** As pointed out above, each weighted projective space appearing as a potential degeneration of $\mathbb{P}^n$ gives a potential degeneration of $\mathbb{P}^{n+1}$: if $\mathbb{P}(a_0, a_1, \ldots, a_n)$ satisfies

$$(n + 1)^n \Pi a_i = \left( \sum a_i \right)^n,$$

then $b = (\Pi a_i)^{1/n} = \sum a_i / n! \in \mathbb{Z}$ and $\mathbb{P}(a_0, a_1, \ldots, a_n, b)$ satisfies

$$(n + 2)^n b \Pi a_i = (b + \sum a_i)^{n+1}.$$

Therefore, the complexity of the problem solely describing solutions for general $\mathbb{P}^n$ seems likely to grow dramatically as $n$ increases.

Although the discussion so far has been on weighted projective space, even the case of log terminal degenerations of $\mathbb{P}^2$, one obtains these spaces and their smoothings. These are easy to describe in this case: each weighted projective space appearing has at most two isolated singularities, and all smoothings are smoothings of one of these points.

In the three-dimensional case, the weighted projective spaces already have non-isolated singularities, so the smoothings are more difficult to describe. For example, they need not be $\mathbb{Q}$-factorial.

**Example 4.44.** Let $X$ be the cone over the anticanonically embedded $\mathbb{P}^1 \times \mathbb{P}^1$. In other words, $X$ is the cone over the Veronese embedding of the quadric surface in $\mathbb{P}^3$. By construction, $X$ is a hyperplane section of $\mathbb{P}(1, 1, 1, 1, 2)$, the cone over the Veronese embedding of $\mathbb{P}^3$. However, we could apply the same construction to the cone over the Veronese embedding of the singular quadric $(xy - z^2 = 0) \subset \mathbb{P}^3$ to realize the cone $\mathbb{P}(1, 1, 2, 4)$ as another hyperplane section of $\mathbb{P}(1, 1, 1, 1, 2)$. Taking an appropriate pencil of these hyperplanes, we realize $X$ as a $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}(1, 1, 2, 4)$.

5. The moduli space

5.1. **Singularities and boundedness.** In the consideration of moduli of pairs $(X, D)$, we can use the classification in the previous section to show that there are no such strictly slc varieties with $d$ odd.

First note that if $(X, D)$ is an H-\(\epsilon\) stable pair of degree $d$ and $(X', \Delta + D')$ the normalization, then the locus of log canonical singularities of $X'$ cannot be contained in $\text{Supp} \Delta + D'$. Using this observation and the work from Section 4, we can prove the following theorem.

**Theorem 5.1.** If $(X, D)$ is an H-\(\epsilon\) stable pair of degree $d$, and $d$ is odd, then $(X, \frac{2}{3} D)$ is semi log terminal.

**Proof.** This is a consequence of the proof of Theorem 4.18. It follows directly from Theorem 4.18 if $X$ is normal, and can be obtained for non-normal $X$ by considering the normalization $(X', \Delta + D')$ and running the same argument, including $\Delta$ in the components of $E$.

This has a number of interesting consequences. First, a corollary of Theorem 4.1:

**Corollary 5.2.** For $d$ odd, the normal varieties $X$ occurring as a degeneration of $\mathbb{P}^3$ in an H-\(\epsilon\) stable pair of degree $d$ are rational.

Secondly, if the $X$ is not normal and the double locus $\Delta$ on $X'$ has more than one component, $\Delta$ must be connected by [ko92, Theorem 17.4]. However, this means $(X', \Delta + D')$ is strictly log canonical, hence we have the following result.
Corollary 5.3. If \( d \) is odd, the varieties \( X \) occurring in \( H\)-\( \epsilon \)-stable pairs of degree \( d \) have at most two components.

As mentioned at the beginning of Section 4, we are also interested in the boundedness of families of \( H\)-\( \epsilon \) stable pairs. For fixed \( \epsilon \), the family of \( H\)-\( \epsilon \) stable pairs is bounded, but allowing \( \epsilon \) to be arbitrary allows one to show that families of \( H\)-\( \epsilon \) stable pairs over a punctured base can be completed in a unique way (Theorem 3.9). However, in light of Theorem 5.1, we can say something about boundedness. The following theorem is a special case of [HMX14a, Corollary 1.7] for threefolds.

Theorem 5.4. The set of \( H\)-\( \epsilon \) stable pairs \( (X, D) \) of fixed odd degree \( d \) form a bounded family.

Proof. Restricting to the normal case, by Theorem 3.1, \( (X, \frac{d}{3}D) \) is klt, and because of the assumption that \( dK_X + 4D \sim 0 \), the pairs \( (X, \frac{d}{3}D) \) are \( \epsilon \)-log terminal because \( dK_X + 4D \sim 0 \) is linear equivalence (not just numerical). Then, \( -K_X \) is ample and \( K_X + \frac{d}{3}D \) is numerically trivial by assumption, hence by [HMX14a, Corollary 1.7], form a bounded family.

We can restrict ourselves to the normal case because the non-normal pairs are in bijection with normal pairs and a certain involution as in [Kol13, Theorem 5.13].

5.2. Algebraicity. Using recent work, we can further analyze the structure of the moduli space of \( H\)-\( \epsilon \) stable pairs. We study Kollár families of pairs, \( \mathbb{Q} \)-Gorenstein families where \( \omega_X^{[n]} \) commutes with base change for all \( n \).

There seem to be more than one avenue to show that the moduli space of \( H\)-\( \epsilon \) stable pairs is an algebraic stack. Using [Hac04, c.f. Theorem 4.4] and following his work, one can show that the moduli space is indeed an algebraic stack.

Definition 5.5. Let \( p \in X \) be a germ of an slc variety. Define the index of \( p \) in \( X \) to be the minimal \( N > 0 \) such that \( NK_X \) is Cartier. Let \( Z \to X \) be the canonical covering \( Z = \text{Spec}_X \mathcal{O} \oplus \mathcal{O}(K_X) \oplus \cdots \oplus \mathcal{O}((N-1)K_X) \), a \( \mu_N \) quotient (c.f. [Rei87]). A deformation \( \mathcal{X}/S \) of \( X \) is \( \mathbb{Q} \)-Gorenstein if there is a \( \mu_N \)-equivariant deformation \( Z/S \) of \( Z \) whose quotient is \( \mathcal{X}/S \).

Definition 5.6. Let \( \mathcal{X}/S \) be a flat family of slc varieties. We say that \( \mathcal{X} \) is weakly \( \mathbb{Q} \)-Gorenstein if, for some \( N > 0 \), \( \omega_{\mathcal{X}/S}^{[N]} \) is invertible. The minimal such \( N \) is called the index of \( \mathcal{X} \).

The following lemmas show that \( \mathbb{Q} \)-Gorenstein implies weakly \( \mathbb{Q} \)-Gorenstein and that the conditions are equivalent if the general fiber is canonical and the base is a curve.

Lemma 5.7. Let \( p \in X \) be a germ of an slc variety. A \( \mathbb{Q} \)-Gorenstein deformation \( \mathcal{X}/S \) of \( X \) of index \( N \) is weakly \( \mathbb{Q} \)-Gorenstein of index \( N \).

Proof. This follows directly from [Hac04, Lemma 3.3].

Lemma 5.8. Let \( \mathcal{X}/T \) be a flat family of slc varieties over the germ of a curve. If the general fiber has canonical singularities and \( K_X \) is \( \mathbb{Q} \)-Cartier, then \( \mathcal{X}/S \) is a \( \mathbb{Q} \)-Gorenstein deformation of \( \mathcal{X}_0 \).

Proof. Using the stronger inversion of adjunction result in [Pat16, Lemma 2.10], the proof of Lemma 3.4 in [Hac04] applies directly.

Many properties of \( \mathbb{Q} \)-Gorenstein deformations are collected in [Hac04, Section 3]. In fact, \( \mathbb{Q} \)-Gorenstein deformations \( X \) are exactly the deformations \( \mathcal{X} \) of \( X \) satisfying the Kollár condition that \( \omega_{\mathcal{X}}^{[n]} \) commutes with base change for all \( n \) [Hac12, 2.4].

However, the presence of the divisor \( D \) can cause further obstructions to deforming \( X \). In particular, taking the canonical cover \( Z \) of \( X \), the associated divisor \( D_Z \) does not have to be a Cartier divisor. This contrasts the picture for plane curves: in [Hac04, Theorem 3.12], it is shown that \( D_Z \) is always Cartier.
In order to avoid obstructions coming from the divisor $D$, we consider higher index covers. By considering the canonical covering of slightly higher index, we can show that studying deformations of the pair $(X, D)$ amounts to studying deformations of $X$ because the presence of the divisor $D$ does not add any further obstructions. In fact, this can be done by taking the canonical covering $Z$ corresponding to $4NK_X$, multiplying the index by 4: the relationship $dK_X + 4D \sim 0$ implies that $D_Z$ is Cartier on $Z$. We will call such a deformation a $4 - \mathbb{Q}$-Gorenstein deformation.

**Theorem 5.9.** Let $(\mathcal{X}, \mathcal{D})/A$ be a $\mathbb{Q}$-Gorenstein family $H$-$\epsilon$ stable pairs. Let $A' \to A$ be an infinitesimal extension and $\mathcal{X}' \to A'$ a $4 - \mathbb{Q}$-Gorenstein deformation of $\mathcal{X}/A$. Then, there exists a $\mathbb{Q}$-Gorenstein deformation $(\mathcal{X}', \mathcal{D}')/A'$ of $(\mathcal{X}, \mathcal{D})/A$.

**Proof.** Using the following lemma in place of Lemma 3.14 in the proof of Theorem 3.12 in [Hac04], the same proof holds. □

**Lemma 5.10.** Let $(X, D)$ be an $H$-$\epsilon$ stable pair. Then, $H^1(X, \mathcal{O}_X(D)) = 0$.

**Proof.** By Serre duality, $H^1(X, \mathcal{O}_X(D)) = H^2(X, \mathcal{O}_X(K_X - D))^\vee$, and $-(K_X - D)$ is ample. If $X$ is log terminal, this follows from Kodaira vanishing. If $X$ is log canonical, we can use a version of Kodaira vanishing in [Fuj15, Theorem 1.2] to get the same conclusion. If $X$ is not normal, we can use an even stronger version of Kodaira vanishing in [KSS10, Corollary 1.3] to conclude the same thing. This result assumes that $X$ is Cohen-Macaulay, but this is automatic by [KK10, Corollary 7.13] for $X$ in an $H$-$\epsilon$ stable pair because $X$ necessarily admits a smoothing to $\mathbb{P}^3$. □

Next, in the definition of $H$-$\epsilon$-stable pairs, we require that $(X, D)$ has a smoothing to $(\mathbb{P}^3, S)$. Therefore, we are interested only in certain ‘smoothable’ deformations of $(X, D)$, made precise below.

**Definition 5.11.** Let $(X, D)/\mathbb{C}$ be an $H$-$\epsilon$ stable pair of degree $d$. Let $(\mathcal{X}^u, \mathcal{D}^u)/S_0$ be a versal $\mathbb{Q}$-Gorenstein deformation of $(X, D)$, where $S_0$ is finite type over $\mathbb{C}$.

Let $S_1 \subset S_0$ be the open subscheme where the fibers of $\mathcal{X}^u$ over $S_0$ are isomorphic to $\mathbb{P}^3$ and $S_2$ the (scheme-theoretic) closure of $S_1$ in $S_0$. A $\mathbb{Q}$-Gorenstein deformation of $(X, D)$ is said to be smoothable if it can be obtained by pullback from the deformation $(\mathcal{X}^u, \mathcal{D}^u) \times_{S_0} S_2 \to 0 \in S_2$.

**Remark 5.12.** Even for degree 5, the moduli space of objects that satisfy the numerical conditions needed to be a stable pair (without admitting a smoothing to $\mathbb{P}^3$) has at least two irreducible components. This condition restricts us to just one component.

**Definition 5.13.** Let $\text{Sch}$ be the category of noetherian schemes over $\mathbb{C}$. For $d \in \mathbb{N}$, we define the stack $\mathcal{M}_d \to \text{Sch}$ by

$$\mathcal{M}_d(S) = \left\{ (X, D)/S \mid (X, D)/S \text{ is a } \mathbb{Q}\text{-Gorenstein smoothable family of } H\text{-}\epsilon \text{ stable pairs of degree } d \right\}$$

As stated above, in [Hac04], it is shown that this $\mathbb{Q}$-Gorenstein deformation condition is equivalent to requiring the Kollár condition on families.

Using the deformation theory in [Hac04, Section 3] and Theorem 3.9, we deduce the following theorem.

**Theorem 5.14.** The moduli space of $H$-$\epsilon$ stable pairs of odd degree $d$ is a proper Deligne-Mumford stack.

**Remark 5.15.** At present, the structure of the proof is:

$H$-$\epsilon$ stable pairs of odd degree have at worst slt singularities
H-\(\epsilon\) stable pairs of odd degree at \(\epsilon\)-slt

\[\downarrow\]

H-\(\epsilon\) stable pairs of odd degree form a bounded family

\[\downarrow (\ast)\]

\(\mathcal{M}_d\) is a proper Deligne Mumford stack

The proof of properness does not rely on oddness of degree; in fact, the only part where oddness of degree is necessary is in the proof of boundedness. Therefore, if we were able to show boundedness in another way for even degree pairs, the rest of the proof is already complete and we would know \(\mathcal{M}_d\) is a proper DM stack for all degree \(d\).

**Remark 5.16.** As mentioned in Section 3, one could remove the condition that H-\(\epsilon\) stable pairs admit a smoothing to \(\mathbb{P}^3\) and define an analogous moduli functor \(\tilde{\mathcal{M}}_d\) of pairs satisfying the first three conditions to be stable and replacing the last condition with

- \(K^3_X = K^3_{\mathbb{P}^3} = -64, \chi(X, \mathcal{O}_X) = \chi(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3})\), and \(X\) is Cohen-Macaulay.

If the functor parameterizes pairs \((X, D)\) belonging to \(\mathcal{Q}\) Gorenstein families, this moduli space is still proper. Although the proof of properness was given for pairs that admit a smoothing to \(\mathbb{P}^3\), the same proof applies more generally. Furthermore, there are still has no strictly log canonical pairs appearing for odd degree \(d\), so we obtain the same theorem:

**Theorem 5.17.** For odd degree \(d\), the moduli space \(\tilde{\mathcal{M}}_d\) is a proper Deligne-Mumford stack.

We will explore \(\mathcal{M}_5\) and \(\tilde{\mathcal{M}}_5\) in Section 6.

One could define an alternative moduli functor via the work of Abramovich and Hassett [AH11].

We can consider the substack of the algebraic stack \(K^{\text{slc}}_{\text{G}}\) (cf. [AH11, Section 5]) satisfying the locally closed condition \(dK_X + 4D \sim 0\) [Kov09, Lemma 5.8]. This condition is algebraic, so we could define a variant \(\mathcal{M}'_d\) of \(\mathcal{M}_d\) as this substack. However, it is not clear if the presence of the divisor \(D\) has an effect on the structure of this stack.

In the future, we hope to explicitly determine the boundary for H-\(\epsilon\) stable pairs of degree 5. Partial progress is described in the next section.

6. The case of quintic surfaces

Because \(\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(5))) = \mathbb{P}^{55}\) and \(\dim \text{Aut} \mathbb{P}^3 = 15\), we have a 40-dimensional space of quintic surfaces in \(\mathbb{P}^3\).

However, just fixing numerical invariants, we obtain a moduli space of smooth quintic surfaces with an additional component [Hor73]. Smooth quintic surfaces have numerical invariants \(K_S^3 = 5, p_g = 4,\) and \(q = 0\) and the moduli space parameterizing these surfaces has two 40 dimensional components.

The first component, consisting of type I surfaces, parameterizes ‘real’ quintic surfaces \(S\) such that \(K_S\) is very ample and defines an embedding \(S \subset \mathbb{P}^3\). The second component parameterizes type IIa surfaces such that \(|K_S|\) has a base-point and \(S\) admits a generically two-to-one morphism to \(\mathbb{P}^1 \times \mathbb{P}^1\). The two components meet along a divisor of dimension 39 parameterizing type IIb surfaces such that \(|K_S|\) has a base-point and \(S\) admits a generically two-to-one morphism to \(\mathbb{F}_2\). For an image of the moduli space and the construction of type II surfaces, see [Ran17].

One might naturally ask how the moduli space of pairs defined in this paper encompasses surfaces of type II.

Surfaces of type IIa cannot appear in \(\mathcal{M}_5\) because we are restricting to pairs that admit smoothings to \(\mathbb{P}^3\) and surfaces in \(\mathbb{P}^3\), but they can indeed appear in \(\mathcal{M}_5\).

To describe them, we recall how to embed surfaces of type II into weighted projective spaces, worked out in [Gri85].
Theorem 6.1 (Griffin). Let $S$ be a numerical quintic surface of type II. Then,

$$S = \mathbb{P}(1, 1, 1, 1, 2, 3, 3)/I$$

where $\mathbb{P}(1, 1, 1, 1, 2, 3, 3)$ has coordinates $(x_0, x_1, x_2, x_3, y, z_1, z_2)$ and $I$ is generated by the relations

\begin{align*}
& r_1 : x_1x_3 - x_2^2 \\
& r_2 : x_1y - (x_2 + \beta x_0)(x_3^2 + \gamma x_0x_3 + \delta x_0^2) \\
& r_3 : (x_2 - \beta x_0)y - x_3(x_3^2 + \gamma x_0x_3 + \delta x_0^2) \\
& r_4 : x_1z_2 - (x_2 + \beta x_0)z_1 \\
& r_5 : (x_2 - \beta x_0)z_2 - x_3z_1 \\
& r_6 : z_1y - z_2(x_3^2 + \gamma x_0x_3 + \delta x_0^2) \\
& r_7 : z_1^2 - \lambda yz_1^2 - x_1Q(x_i, y) - x_0e_1 \\
& r_8 : z_1z_2 - \lambda y^2z_3 - x_2Q(x_i, y) - x_0e_2 \\
& r_9 : z_2^2 - \lambda y^3 - x_3Q(x_i, y) - x_0e_3
\end{align*}

where $Q$ and $e_i$ are weight 5 polynomials satisfying certain conditions. The surface $S$ is of type IIb if $\beta = 0$ and type IIa if $\beta \neq 0$.

We begin with the simplest example: $\beta = \gamma = \delta = \lambda = e_i = 0$. In this case, we will show that $S$ is a hypersurface of degree 50 on $X = \mathbb{P}(1, 4, 10, 25)$, so it satisfies $5K_X + 4S \sim 0$.

Example 6.2. Let $X = \mathbb{P}(1, 4, 10, 25)$ with coordinates $a_0, a_1, a_2, a_3$. First, consider the embedding

$$X \to \mathbb{P}(1, 2, 5, 13, 25)$$

given by

$$(a_0, a_1, a_2, a_3) \mapsto (a_0^2, a_1, a_2, a_0a_3, a_3^2)$$

so that, if $\mathbb{P}(1, 2, 5, 13, 25)$ has coordinates $b_0, b_1, b_2, b_3, b_4$, then

$$X : (b_0b_3 - b_4^2 = 0) \subset \mathbb{P}(1, 2, 5, 13, 25).$$

Then, consider the embedding

$$\mathbb{P}(1, 2, 5, 13, 25) \to \mathbb{P}(1, 1, 1, 1, 2, 3, 3, 5)$$

given by

$$(b_0, b_1, b_2, b_3, b_4) \mapsto (b_2, b_0^5b_1, b_0b_1^2, b_1^5b_3, b_1b_3, b_4).$$

If $\mathbb{P}(1, 1, 1, 1, 2, 3, 3, 5)$ has coordinates $x_0, x_1, x_2, x_3, y, z_1, z_2, t$, in the composition

$$X \to \mathbb{P}(1, 1, 1, 1, 2, 3, 3, 5)$$

we find that $X$ is defined by the equations

\begin{align*}
& r_1 : x_1x_3 - x_2^2 \\
& r_2 : x_1y - x_2x_3^2 \\
& r_3 : x_2y - x_3^3 \\
& r_4 : x_1z_2 - x_2z_1 \\
& r_5 : x_2z_2 - x_3z_1 \\
& r_6 : z_1y - z_2x_3^2 \\
& r_7 : z_1^2 - x_1t \\
& r_8 : z_1z_2 - x_2t \\
& r_9 : z_2^2 - x_3t
\end{align*}

Comparing these to the equations in Theorem 6.1 for $\beta = \gamma = \delta = \lambda = e_i = 0$, the only difference is that $t = Q(x_i, y)$. Therefore, let $S$ be the surface $t = Q(x_i, y)$ in $\mathbb{P}(1, 1, 1, 1, 2, 3, 3, 5)|_X$. As desired, $S$ has degree 50 on $X$. 

Furthermore, when $\beta = 0$, the surface $S$ defined by the equation in Theorem 6.1 are of type IIb, so do admit smoothings to pairs $(\mathbb{P}^3, S)$. That is the case in this example as we can smoothing $\mathbb{P}(1, 4, 10, 25)$ to $\mathbb{P}^3$ and bring the surface along.

Similar to the previous example, by using various smoothings of $\mathbb{P}(1, 4, 10, 25)$, we can obtain the general surfaces of type II. For example, if we consider a partial smoothing $Y_{26}$ of $\mathbb{P}(1, 4, 10, 25)$ in $\mathbb{P}(1, 2, 5, 13, 25)$ given by $b_0b_3 - b_2^2 = f_2(\theta)$ and its image in $\mathbb{P}(1, 1, 1, 1, 2, 3, 3, 5)$ we can find surfaces $t = Q(x_i, y)$ on $Y$ where $\lambda \neq 0$ and $e_i \neq 0$.

In general, this is complicated to write down because the threefolds $S$ containing the surfaces of type II are not complete intersections even in weighted projective space.

### 6.1. A divisor in the moduli space and components of higher codimension

Returning to $\mathcal{M}_5$, we study divisors and components of higher codimension.

Let $X$ be the cone over the Veronese embedding of the quadric surface. This is a cone over $\mathbb{P}^1 \times \mathbb{P}^1$ and is a toric variety, c.f. Example 4.20. A toric computation shows that the projective dimension of $\text{Aut} X$ is 15.

If $X$ appears in an $H$-stable pair of degree 5, we must have $5K_X + 4D \sim 0$. Because $X$ is a section of $\mathcal{O}_W(2)$ for the weighted projective space $W = \mathbb{P}(1, 1, 1, 1, 2)$, we compute that $\mathcal{O}_X(K_X) = \mathcal{O}_W(-4)|_X$ and generic $D$ satisfying $5K_X + 4D \sim 0$ is a section of $\mathcal{O}_X(D) = \mathcal{O}_W(1)|_X$. The projective dimension of the linear system $|\mathcal{O}_X(D)|$ is 54, hence the dimension of the space of pairs $(X, D)$ is 39. This locus $D$ has codimension 1 in the moduli space, so one might ask if it forms a known divisor in the space.

For general $D \in |\mathcal{O}_W(1)|_X$, because $D$ is a complete intersection in $\mathbb{P}(1, 1, 1, 1, 2)$, we can compute the singularities as in [IF00, Section 1.7]. The computation shows that $D$ has a unique $\frac{1}{4}(1, 1)$ singularity at the vertex of $X$.

Furthermore, studying the $4 - \mathbb{Q}$-Gorenstein deformations of $X$ allows us to conclude that $\mathcal{M}_5$ is smooth at the generic point of $D$. Indeed, one first computes the $4 : 1$ cyclic cover of $Z \to X$ as in Section 5, and $Z$ is smooth, so the obstructions vanish.

One can compare this with existing work on GIT for quintic surfaces. In [Ran17, Theorem 1.5], it is shown that there is a divisor $D'$ parameterizing surfaces whose unique non Du Val singularity is $\frac{1}{4}(1, 1)$. The component $D$ in $\mathcal{M}_5$ here parameterizes the surfaces Rana calls ‘type 1’ (appearing as a divisor on the component parameterizing surfaces of type I). In other words, for general $S$ such that $[S] \in D'$ in Rana’s work, $S$ appears as a divisor on the threefold $X$ where $[(X, S)] \in D$ in this interpretation.

Pressing further, from Example 4.44, we know that $\mathbb{P}(1, 1, 2, 4)$ admits a smoothing to $X$, so should correspond to a higher codimension piece of $\mathcal{M}_5$. Indeed, a toric computation shows that the projective dimension of the automorphism group of $\mathbb{P}(1, 1, 2, 4)$ is 17, and surfaces on $Z = \mathbb{P}(1, 1, 2, 4)$ satisfying $5K_Z + 4D \sim 0$ are elements of the linear system $|\mathcal{O}_Z(10)|$. This linear system has projective dimension 55, so the space parameterizing surfaces on $\mathbb{P}(1, 1, 2, 4)$ has dimension 38. This is a codimension 2 component of $\mathcal{M}_5$ that is codimension 1 inside $D$. Furthermore, a computation as in [IF00, Section 1.7] shows that the surfaces appearing on $\mathbb{P}(1, 1, 2, 4)$ have two singularities: $\frac{1}{4}(1, 1)$ and $\frac{1}{2}(1, 1)$. We could continue further: there is a 37-dimensional (or codimension 3) component parameterizing surfaces on $\mathbb{P}(1, 2, 9, 12)$. This admits a smoothing to $\mathbb{P}(1, 1, 2, 4)$ (see Section 4.3) and the surfaces appearing on $\mathbb{P}(1, 2, 9, 12)$ have an additional $\frac{1}{4}(1, 2)$ singularity.

We can also describe some non-normal threefolds appearing using Propositions 4.36 and 4.40. For instance, considering the mutations going from $(1, 1, 1, 1)$ to $(1, 1, 2, 4)$, from $(1, 1, 2, 4)$ to $(1, 4, 10, 25)$, and from $(1, 1, 2, 4)$ to $(1, 2, 9, 12)$, we obtain the three non-normal threefolds $\mathbb{P}(1, 1, 1, 2) \cup \mathbb{P}(1, 1, 1, 2)$, $\mathbb{P}(1, 1, 2, 5) \cup \mathbb{P}(1, 4, 2, 5)$, and $\mathbb{P}(1, 1, 3, 4) \cup \mathbb{P}(1, 2, 3, 4)$.

To explicitly describe all threefolds appearing in $\mathcal{M}_d$ or $\hat{\mathcal{M}}_d$, we must complete the classification begun in Section 4.3. There is more work to be done, but it seems as though one can recover information in the GIT moduli space of quintic surfaces with this description. Furthermore, if we
denote by $\mathcal{M}_d^{GIT}$ the GIT moduli space, one expects a rational map

$$\mathcal{M}_d \rightarrow \mathcal{M}_d^{GIT}$$

although understanding this map would require a better understanding of both $\mathcal{M}_d$ and $\mathcal{M}_d^{GIT}$.

References


MODULI OF HYPERSURFACES IN $\mathbb{P}^3$


