Lecture Notes: Math 18 Weeks 0 & 1

September 30, 2019

1 1.1: Systems of Linear Equations

Let's motivate linear algebra with an example.

Example 1.1. Amelia is two years older than Reuben. Their ages sum to 6. How old are Amelia and Reuben?

$$\begin{aligned} x + y &= 6 \\ x - y &= 2. \end{aligned}$$

There are three ways to think about this: substitution, doing "algebra" with equations, or graphically. Many of you are probably used to solving this with substitution. For example, if we solve the second equation for x, we find that

$$x = y + 2.$$

Now, we can plug this into the first equation to get

$$y + 2 + y = 6$$

which means $y = 2$, and $x = 4$. There is nothing wrong with this method, but we'll soon be trying to solve more complicated systems (with lots of variables and lots of equations), so we might want to focus on a different method.

We could solve this equation in another way by doing a little "equation algebra." For instance, if we just add the first equation to the second, we get

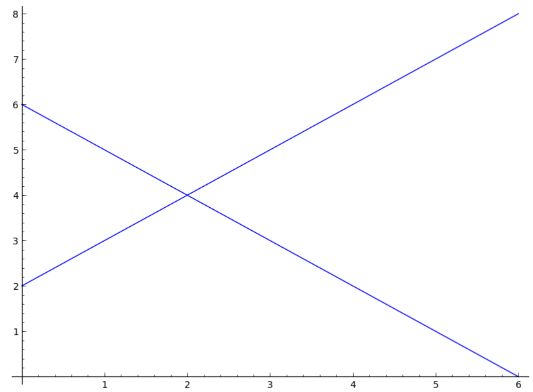
$$x + y + x - y = 6 + 2$$

or

$$2x = 8$$

which immediately tells us that $x = 4$. We can then solve for y to see $y = 2$ as well. This is the process we are going to formalize in this course.

Finally, another beneficial way to do this would be to graph the linear system. The solution is the intersection of two lines. We know how to graph lines and can just sketch them to find the solution. We'll come back to this throughout the course.



What if we wanted to solve a harder question? We'll come back to this later. The course will be focused on studying these equations and systems of these equations. Let's begin with the terminology.

Notation 1.2. Rather than 'x' and 'y' we might have a lot of variables. So we use $x_1, x_2, \ldots, x_n$ denote *different* variables. Other ways they might appear are as x_i, y_i, w_i, z_i. We also let $a_1, \ldots, a_n \in \mathbb{R}$ denote constants. Here $a_i \in \mathbb{R}$ means a_i is an element of the set of real numbers.

Definition 1.3. A **linear equation** is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b \in \mathbb{R}$ are constants and $x_1, x_2, \ldots, x_n$ are variables.

Example 1.4.

$$2x_1 + 3x_2 + x_3 = 6$$

is an example of a linear equation with 3 variables.

Example 1.5. $2x_1^2 + 3x_1x_3 = 7$ is not a linear equation. Neither is $x_1/x_2 = 6$.

Definition 1.6. A **solution** to a linear equation is a set of numbers $(s_1, s_2, \ldots, s_n)$ that satisfies the equation.

Example 1.7. By plugging in $(1, 1, 1)$ to the equation

$$2x_1 + 3x_2 + x_3 = 6,$$

we get

$$2 + 3 + 1 = 6$$

which shows that $(1, 1, 1)$ is a solution to the equation.

Definition 1.8. The **solution set** to a linear equation is the set of all solutions to a given equation.

Definition 1.9. A **system of linear equations** or **linear system** is a collection of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Notice the double subscripts; a_{ij} means ith down and jth over when arranged in standard form.

The following is an example of a linear system with 3 variables and 3 equations.

Example 1.10.

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 6 \\ x_1 - x_2 - x_3 &= 2 \\ 2x_1 + x_2 - 7x_3 &= 1 \end{aligned}$$

Definition 1.11. A **solution** to a linear system is a set of numbers $(s_1, s_2, \ldots, s_n)$ that simultaneously satisfy all of the equations in the linear system.

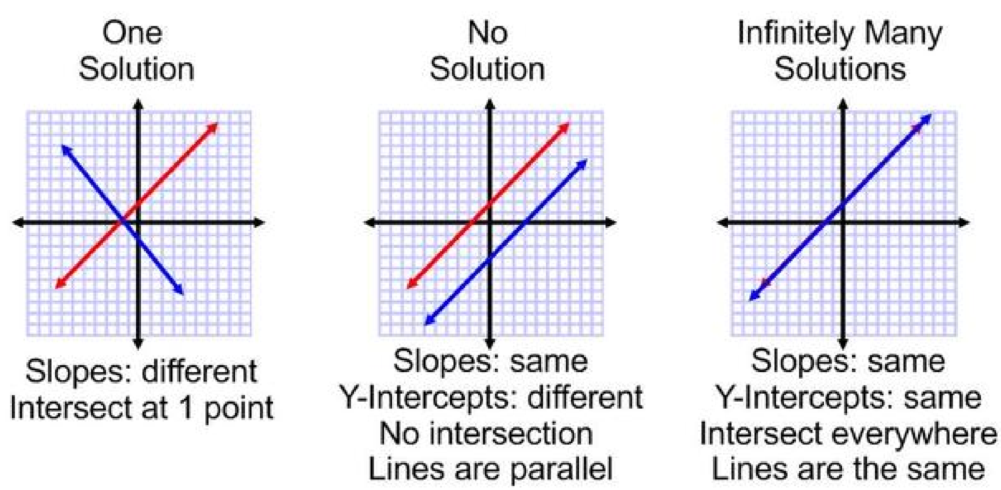
Definition 1.12. We say a linear system is **consistent** if it has at least one solution. We say it is **inconsistent** if it has no solutions.

Keep in mind that there can be more than one solution to a linear system; for example the system

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ 4x_1 + 6x_2 &= 12 \end{aligned}$$

has infinitely many solutions (because it is two multiples of the same equation).

We can also use the graph perspective to analyze solution sets.



In the example of two lines intersecting (solving a system of two equations with two variables), we know the lines could either intersect in one point (meaning there is exactly one solution), or intersect nowhere by being parallel (meaning there are no solutions), or be the same line (meaning there are infinitely many solutions). This is our first indication of the following theorem, which we'll prove soon.

Theorem 1.13. A linear system has no solutions, exactly one solution, or infinitely many solutions.

Two questions: Why is linearity important here? And, how do we actually find these solutions? We want to develop an algorithm to systematically find the solutions. First, we need some notation.

2 1.2: Linear Systems and Matrices

In this section, we'll figure out how to easily find solutions. To do that, we're going to introduce *matrices*.

Definition 2.1. A **matrix** is a rectangular array of numbers.

Example 2.2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & -1 & 2 \\ 1 & 4 & 7 \end{pmatrix}$$

are examples of matrices.

Notation 2.3. We will use both hard $[\cdot]$ and soft $(\cdot)$ brackets for matrices. They both mean the same thing.

Each linear system corresponds to two matrices.

Definition 2.4. The linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

has **coefficient matrix**

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

It has **augmented matrix**

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & \ddots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

For the first few chapters, we'll focus on the augmented matrix only. Note that the book leaves out the vertical line separating the a and b terms.

Example 2.5. The linear system

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &= 2 \\ x_1 - x_2 + 4x_3 &= -1 \end{aligned}$$

has augmented matrix

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 2 \\ 1 & -1 & 4 & -1 \end{array}\right].$$

Example 2.6. Let's go back to the first example we tried to solve, with equations

$$\begin{aligned} x_1 + x_2 &= 6 \\ x_1 - x_2 &= 2. \end{aligned}$$

This has augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 1 & -1 & 2 \end{array}\right].$$

Using "equation algebra" we saw that we could add and subtract equations to solve the system. For example, say we take the second equation and subtract the first equation from it. This is particularly simple to do using the matrix: we take the second row of the matrix and subtract the first row from it column by column:

$$\left[\begin{array}{cc|c} 1 & 1 & 6 \\ 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 1-1 & -1-1 & 2-6 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 6 \\ 0 & -2 & -4 \end{array}\right].$$

Looking at just the last row, this says that $-2x_2 = -4$, and dividing that by -2, we get $x_2 = 2$. Plugging this into the first equation says $x_1 = 4$, as we found before.

These "equation algebra" steps have names.

Definition 2.7. Elementary row operations are operations that we can do to a linear system that don't change its solution set. There are three of them:

1. Interchange two rows of the matrix
2. Multiply one row by a nonzero constant
3. Add a multiple of one row to another row

We can use these row operations and matrices to systematically find solutions to systems of equations. It is important to note that all of these operations are *reversible*, which means that doing them does not change the solution to the linear systems. We need to introduce some more terminology.

Definition 2.8. A matrix is in **echelon form** if

(a) the leading terms (the first nonzero entries of each row) move down and to the right in a stair step pattern.

(b) any rows of zeros are grouped at the bottom of the matrix.

$$\left(\begin{array}{ccc|c} 1 & -\frac{7}{10} & -2 & -\frac{2}{5} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Definition 2.9. A matrix is in **reduced echelon form** if

(a) it is in echelon form

(b) all leading terms are equal to 1

(c) each leading term is the only nonzero term in its column

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, we will outline an algorithm for converting a matrix into an equivalent matrix in reduced echelon form, and see how that gives us the solutions to the corresponding linear system. We will outline the process with an example.

Example 2.10. Row reduction. Consider the system

$$\begin{aligned} 2x_1 - 4x_2 - 6x_3 &= -2 \\ x_1 - x_2 - 2x_3 &= 1 \\ -x_1 + 3x_2 + 5x_3 &= 2. \end{aligned}$$

1. Write the linear system as the corresponding augmented matrix.

   In the example,

$$\left[\begin{array}{ccc|c} 2 & -4 & -6 & -2 \\ 1 & -1 & -2 & 1 \\ -1 & 3 & 5 & 2 \end{array}\right].$$

2. If necessary, interchange two rows to get a nonzero entry in the upper left corner. Multiply the first row by a constant to make that entry a 1.

   In the example, this means multiplying the first row by -1/2,

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 1 & -1 & -2 & 1 \\ -1 & 3 & 5 & 2 \end{array}\right].$$

3. Add multiples of the first row to the other rows to zero out the rest of the column.

   In the example, this corresponds to adding $-R_1$ to R_2 and adding R_1 to R_2,

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right].$$

4. Move over one column and down one row and repeat.

   In this example, we move to the entry in the second row and second column. It's already a 1, so we proceed to step 3 to zero out the rest of the second column by adding $2R_2$ to the first row adding $-R_2$ to R_3,

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

   Now, we move down and over again to the third row and third column. It's also a 1, so we proceed to step 3 and zero out the rest of the column,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

5. Translate back to system of equations and write down the solution. There may be exactly one solution, or there may be free variables or no solutions.

   In the example, if we translate back to a system of equations, we see that

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 3 \\ x_3 &= -1 \end{aligned}$$

   so the solution is $(2, 3, -1)$. It's always a good idea to check this!

This process will work for any system.

Example 2.11. Here's an example where there are infinitely many solutions:

$$\left(\begin{array}{ccc|c} -1 & 3 & 9 & 7 \\ -1 & 2 & 6 & 5 \\ -2 & 7 & 21 & 16 \end{array}\right)$$

$$-1R1$$

$$\left(\begin{array}{ccc|c} 1 & -3 & -9 & -7 \\ -1 & 2 & 6 & 5 \\ -2 & 7 & 21 & 16 \end{array}\right)$$

$$1R1 + R2$$

$$2R1 + R3$$

$$\left(\begin{array}{rrr|r} 1 & -3 & -9 & -7 \\ 0 & -1 & -3 & -2 \\ 0 & 1 & 3 & 2 \end{array}\right)$$

$$-1R2$$

$$\left(\begin{array}{rrr|r} 1 & -3 & -9 & -7 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{array}\right)$$

$$-1R2 + R3$$

$$\left(\begin{array}{rrr|r} 1 & -3 & -9 & -7 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced echelon form:

$$\left(\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Looking at this, how do we write the solution? Because there is no leading one in the x_3, column, there are no constraints on what values x_3 can take, so x_3 is a *free variable*. So, in the solution, we simply say that x_3 is free, and solve for the other variables in terms of x_3, and get that the solution is

$$\begin{aligned} x_1 &= -1 \\ x_2 &= -3x_3 + 2 \\ x_3 &\text{ is free.} \end{aligned}$$

Example 2.12. Here's an example where there are no solutions.

$$\left(\begin{array}{rr|r} 10 & -3 & -5 \\ -3 & 1 & -3 \\ 5 & -2 & 4 \end{array}\right)$$

$$1/10R1$$

$$\left(\begin{array}{rr|r} 1 & -\frac{3}{10} & -\frac{1}{2} \\ -3 & 1 & -3 \\ 5 & -2 & 4 \end{array}\right)$$

$$3R1 + R2$$

$$-5R1 + R3$$

$$\left(\begin{array}{rr|r} 1 & -\frac{3}{10} & -\frac{1}{2} \\ 0 & \frac{1}{10} & -\frac{9}{2} \\ 0 & -\frac{1}{2} & \frac{13}{2} \end{array}\right)$$

$$10R2$$

$$\left(\begin{array}{cc|c} 1 & -\frac{3}{10} & -\frac{1}{2} \\ 0 & 1 & -45 \\ 0 & -\frac{1}{2} & \frac{13}{2} \end{array}\right)$$

$$1/2R2 + R3$$

$$\left(\begin{array}{cc|c} 1 & -\frac{3}{10} & -\frac{1}{2} \\ 0 & 1 & -45 \\ 0 & 0 & -16 \end{array}\right)$$

$$-1/16R3$$

$$\left(\begin{array}{cc|c} 1 & -\frac{3}{10} & -\frac{1}{2} \\ 0 & 1 & -45 \\ 0 & 0 & 1 \end{array}\right)$$

Reduced echelon form:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

What does the last equation say? It says that $0 = 1$! This is not true! So there cannot be any solutions to this linear system.

Definition 2.13. Your book has special terminology about leading ones in reduced echelon form. A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Let's make a few observations about systems in reduced echelon form.

1. In reduced echelon form, if the last row looks like $[0 \ \dots \ 0 \mid 1]$, that translates to the equation $0 = 1$, which is FALSE, so the system has NO SOLUTIONS (i.e. is *inconsistent*).
2. If there is not a row like that, then either each column has a leading one (i.e. is a pivot column), meaning there is exactly one solution, or there is at least one column without a leading one (i.e. is not a pivot column), meaning it is free and can take any value, so there are infinitely many solutions.
3. Combining the two statements above, we've actually proven that linear systems have either 0, 1, or infinitely many solutions, because these are the only possibilities in reduced echelon form.
4. If there are more variables than equations (more columns than rows in the matrix), then each column CANNOT have a leading one. This means the system has either no solutions or infinitely many, but it cannot have exactly one.

There is a special type of linear systems that ALWAYS has a solution.

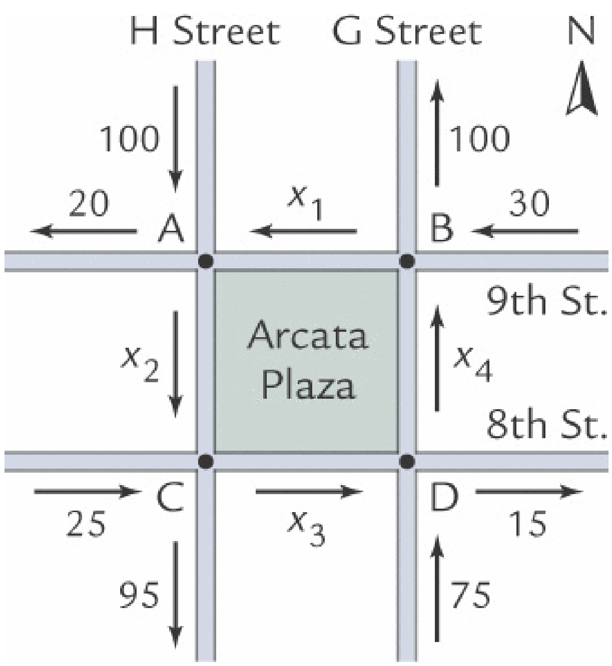
Definition 2.14. A linear system is **homogeneous** if all of the constant terms are equal to 0, i.e. it looks like

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Note that these systems *always* have a solution, namely $(0, 0, \ldots 0)$, called the **trivial solution**. There may be (infinitely many) other solutions, but we at least know the system is consistent.

3 Optional Application: Traffic flow

Arcata, on the northern coast of California, is a small college town with a central plaza (Figure 1). Figure 2 shows the streets surrounding and adjacent to the towns central plaza. As indicated by the arrows, all streets in the vicinity of the plaza are oneway.



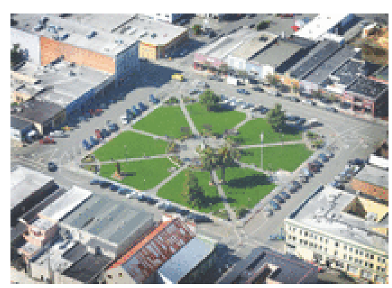


Figure 2 Traffic volumes around the Arcata plaza. Traffic flows north and south on G and H streets, respectively, and east and west on 8th and 9th streets, respectively. The number of cars flowing on and off the plaza during a typical 15-minute period on a Saturday morning is also shown. Our goal is to find x_1, x_2, x_3, and x_4, the volume of traffic along each side of the plaza. The four intersections are labeled A, B, C, and D. At each intersection, the number of cars entering the intersection must equal the number leaving. For example, the number of cars entering A is $100 + x_1$ and the number exiting is $20 + x_2$. Since these must be equal, we end up with the equation

$$A : 100 + x_1 = 20 + x_2$$

Applying the same reasoning to intersections B, C, and D, we arrive at three more equations,

$$B : x_3 + 30 = x_1 + 100$$

$$C : x_2 + 25 = x_3 + 95$$

$$D : x_3 + 75 = x_4 + 15.$$

Rewriting the equations in the usual form, we obtain the system:

$$\begin{aligned} x_1 - x_2 &= -80 \\ x_1 - x_4 &= -70 \\ x_2 - x_3 &= 70 \\ x_3 - x_4 &= -60 \end{aligned}$$

To solve the system, we populate an augmented matrix and transform to echelon form.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -80 \\ 1 & 0 & 0 & -1 & -70 \\ 0 & 1 & -1 & 0 & 70 \\ 0 & 0 & 1 & -1 & -60 \end{array}\right)$$

$$-1R1 + R2$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -80 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 1 & -1 & 0 & 70 \\ 0 & 0 & 1 & -1 & -60 \end{array}\right)$$

$$-1R2 + R3$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -80 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & -1 & 1 & 60 \\ 0 & 0 & 1 & -1 & -60 \end{array}\right)$$

$$-1R3$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -80 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & -1 & -60 \\ 0 & 0 & 1 & -1 & -60 \end{array}\right)$$

$$-1R3 + R4$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -80 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -70 \\ 0 & 1 & 0 & -1 & 10 \\ 0 & 0 & 1 & -1 & -60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The last coordinate x_4 is free. Back substitution yields the general solution

$$x_1 = -70 + x_4, \quad x_2 = 10 + x_4, \quad x_3 = -60 + x_4, \quad x_4 \text{ is free.}$$

A moments thought reveals why it makes sense that this system has infinitely many solutions. There can be an arbitrary number of cars simply circling the plaza, perhaps looking for a parking space. Note also that since each of x_1, x_2, x_3, and x_4 must be nonnegative, it follows that the parameter $x_4 \geq 70$. The analysis performed here can be carried over to much more complex traffic questions, or to other similar settings, such as computer networks.

4 Optional Application: Data fitting

Given a set of ordered pairs we want to find a curve that goes through all of them.

Example 4.1. Find a quadratic through $(0,1), (1,2)$ and $(2,7)$.
We know that a quadratic has equation

$$y = a_1 x^2 + a_2 x + a_3.$$

Plugging in the data points for (x, y) in the equation, we get a linear system in a_1, a_2, a_3:

$$\begin{aligned} a_3 &= 1 \\ a_1 + a_2 + a_3 &= 2 \\ 4a_1 + 2a_2 + a_3 &= 7 \end{aligned}$$

Now, we solve this system:

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 7 \end{array}\right)$$

$$R1 \circlearrowleft R2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 4 & 2 & 1 & 7 \end{array}\right)$$

$$-4R1 + R3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & -2 & -3 & -1 \end{array}\right)$$

$$R2 \circlearrowleft R3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$-1/2R2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Reduced echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

This says that $a_1 = 2, a_2 = -1$, and $a_3 = 1$, so our quadratic has equation $y = 2x^2 - x + 1$.

Example 4.2. Find the plane equation going through $(1, 1, 0)$, $(1, 2, 1)$ and $(0, 0, 1)$.

We do the same thing as in the previous example, but use the equation of a plane:

$$a_1x_1 + a_2x_2 + a_3x_3 - b = 0.$$

Plug in our three points (x_1, x_2, x_3) to get and solve the linear system:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$-1R1 + R2$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

Reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right)$$

$$a_1 = 2b, a_2 = -b, a_3 = b.$$

So we have infinitely many solutions! But what is one solution? Well, set b equal to anything nonzero. For instance, if $b = 1$ then we have

$$2x_1 - x_2 + x_3 - 1 = 0$$

is a plane containing those points. Note that it makes sense to have infinitely many solutions because any nonzero multiple of the equation of a given plane determines the same plane.

5 1.3: Vectors

Definition 5.1. A **vector** is an ordered list of real numbers (called **components**) arranged in a vertical column or in wedge brackets $\langle \cdots \rangle$ or in regular parentheses $(\ldots)$.

Example 5.2.

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \langle 1, 2, 3 \rangle \qquad (1, 2, 3)$$

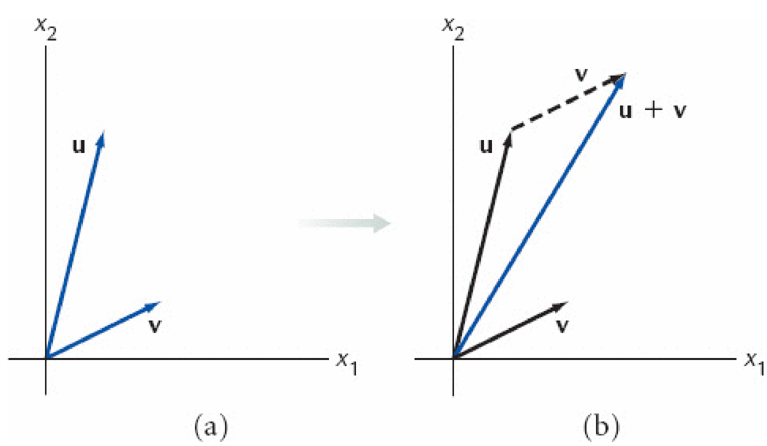
We need to know where vectors live.

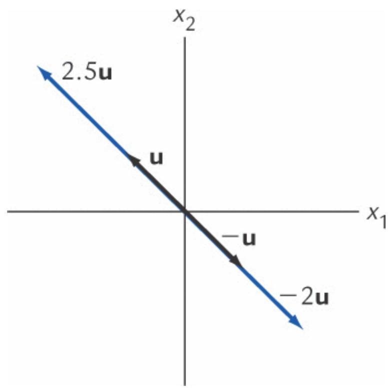
Notation 5.3. We write $a \in \mathbb{R}$ to mean that a is a real number. Often we call a a **scalar**. Vectors, on the other hand, live in higher-dimensional space. We can think of vectors as *points* in higher dimensional space: the vector $(1, 2, 3)$ can be interpreted as just the point $(1, 2, 3)$. With this idea, we think of all n-dimensional vectors as giving us the entire n dimensional space, and the set of all n-dimensional vectors is denoted by $\mathbb{R}^n$. The symbol $\mathbb{R}$ is the set of all real numbers. We write $\vec{u} \in \mathbb{R}^n$ to mean $\vec{u}$ is a vector with n components.

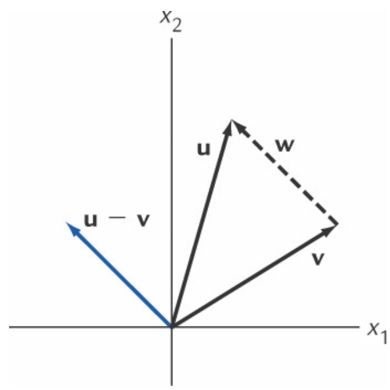
Definition 5.4. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$. That is $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

- $\vec{u} = \vec{v}$ means $u_i = v_i$ for each component.
- $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$
- $c\vec{u} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$

There are visual ways to understand the vector operations:







Example 5.5. $\begin{bmatrix}3\\2\\0\end{bmatrix} + 2\begin{bmatrix}3\\2\\0\end{bmatrix} = \begin{bmatrix}11\\4\\6\end{bmatrix}$

Definition 5.6. A **linear combination** of the vectors $\vec{u}_1, \ldots, \vec{u}_m$ is the vector

$$c_1\vec{u}_1 + \cdots c_m\vec{u}_m$$

for some scalars $c_1, \ldots, c_m$.

5.1 An application: graphics

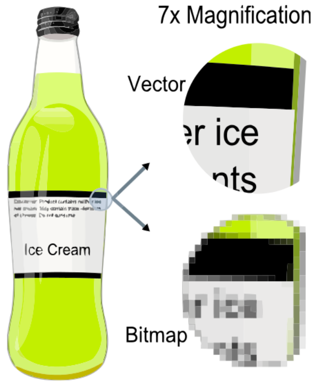
Application 5.7. Vector graphics such as PDF and CAD use vectors to make scalable images. Rather than recording image pixel data, these record everything as either a line, circle, ellipse, curve with start and end points. Rescaling is then multiplying by a constant. So vector graphics can be scaled infinitely! Basically the file has 1000's of instructions that read like:

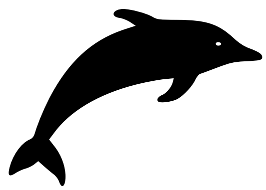
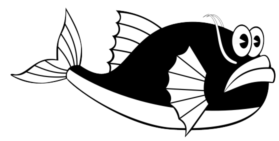
- Points in a vector graphic are described by a vector describing their position from the origin.
- Draw a dot at point 'a'
- Draw a dot at point 'b'
- Connect the points together with a straight line (or curve) of a certain thickness
- Color the line, or fill an enclosed space

All of the letters in this pdf are recorded as vectors!

Advantage: The size of the file does not change if the image needs to be larger—the same instructions will make the image as large as you like, and yet the line would always remain smooth and clear. We'll see later that rotations, reflections and translations are very easy with vectors.

Simple geometric shapes such as letters, autoshapes and logos are ideal for storing as vector images. Some images are not suitable for storing as a vector image. For example, a detailed photograph would need so many vector instructions to recreate it, that it is better to store it as, say, a BITMAP.



Example 5.8. This '→' is stored as a vector graphic consisting of three lines. Each line is stored as a vector and a base point for the vector. Suppose that the base of the arrow is at the origin and has length 4.

Describe the vector and base point of the three lines in the arrow. (There are many choices that could work.) Then, rescale it to be 4 times larger in size. What would make it point left instead of right?

Solution: The line from base to tip is rooted at $(0,0)$ with the vector $\begin{bmatrix}4\\0\end{bmatrix}$. The two points of the arrows are the vectors $\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}$ based at $(4,0)$. To rescale we multiply everything by 4. Multiplying everything by -1 would change the direction.

5.2 Span

Now, we will introduce one of the fundamental concepts in this course. Vectors give us directions, so we can think of them as 'building blocks' in n-dimensional space. The span will be the set of all points we can build with our vectors by only going in the directions they allow. Better yet, think of vectors like your favorite smoothie ingredients. Imagine you want to blend them together to hit a certain nutrient profile (fat, protein, carbs, iron, calcium, ...). Studying the span asks what profiles we can make with the vectors and which ones we cannot. We expand on this here with an example in $\mathbb{R}^2$.

Example 5.9. Imagine we have vectors $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$ and we want to use them to get to the point $(5,6)$. Can we do this? Well, with $\vec{v}$ we can adjust our horizontal position as we please. So, we first use $\vec{u}$ to get to height 6. This is at $3\vec{u} = \begin{bmatrix}3\\6\end{bmatrix}$. Next, we need to shift back by 1 in the x-coordinate. This requires $-\vec{v}$. So

$$3\vec{u} - \vec{v} = \begin{bmatrix}5\\6\end{bmatrix}.$$

In this manner we can express any point in $\mathbb{R}^2$ using $\vec{u}$ and $\vec{v}$. To see this, take any point $(x,y) \in \mathbb{R}^2$ and solve

$$c_1\vec{u} + c_2\vec{v} = \begin{bmatrix}x\\y\end{bmatrix}.$$

This is equivalent to

$$\begin{aligned} c_1 + c_2 &= x \\ 2c_1 &= y \end{aligned}$$

(a system of linear equations that we know how to solve!!!). Solving this, we see that, for any choice of x and y, $c_1 = y/2$ and $c_2 = x - \frac{y}{2}$. This is the recipe to get to any point (x, y) by only going in the directions of $\vec{u}$ and $\vec{v}$.

Example 5.10. What if we take $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and ask where we can go in $\mathbb{R}^3$ with these? Can we get to any point? Well, we can go to any point that is a linear combination of $\vec{u}_1$ and $\vec{u}_2$. Because

$$x_1\vec{u}_1 + x_2\vec{u}_2 = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix},$$

we see that we can only get to vectors

$$[abc]$$

whose coordinates satisfy the system of linear equations

$$\begin{aligned} 2x_1 + x_2 &= a \\ x_1 + 2x_2 &= b \\ x_1 + 3x_2 &= c. \end{aligned}$$

This does not always have a solution! Let's show that we cannot form $(1, 0, 0)$ with these:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 9 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Because this is inconsistent, it shows that we can't make everything with just these two vectors.

Given a set of vectors in $\mathbb{R}^n$ we are interested in the set of all points we can make with linear combinations. This is called the **span**.

Definition 5.11. The **span** of vectors $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$ is the set of all linear combinations

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m,$$

where $c_1, \ldots, c_m \in \mathbb{R}$.

Whether or not a matrix has a solution tells us whether a vector belongs to the span.

Theorem 5.12. Let $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$. Then $v \in \operatorname{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$ if and only if the matrix equation $[\vec{u}_1 \cdots \vec{u}_m \mid \vec{v}]$ has at least one solution.

Also, adding a redundant vector doesn't change the span.

Theorem 5.13. Let $S = \text{span}\{\vec{u}_1, \ldots, \vec{u}_m\}$. If $\vec{u} \in S$ then $S = \text{span}\{\vec{u}, \vec{u}_1, \ldots, \vec{u}_m\}$.

Example 5.14. Is $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^3$?

Call these vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Given $(a, b, c) \in \mathbb{R}^3$ we want to find x_1, x_2, x_3 so that

$$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

This gives matrix

$$\begin{pmatrix} 1 & -1 & 1 & a \\ 1 & 0 & 0 & b \\ 2 & 1 & 0 & c \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & -2\,b + c \\ 0 & 0 & 1 & a - 3\,b + c \end{pmatrix}$$

This is consistent for any choice of a, b, c so the answer is yes!

Example 5.15. Is $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^3$?

Let's setup a matrix and reduce.

$$\begin{pmatrix} 1 & 3 & 2 & a \\ 1 & 1 & 1 & b \\ 1 & -1 & 0 & c \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2}\,a + \frac{3}{2}\,b \\ 0 & 1 & \frac{1}{2} & \frac{1}{2}\,a - \frac{1}{2}\,b \\ 0 & 0 & 0 & a - 2\,b + c \end{pmatrix}$$

This is only consistent if $a - 2b + c = 0$, so the span is not equal to all of $\mathbb{R}^3$! The span is exactly the plane $a - 2b + c = 0$.

The previous example is a case of a more general phenomenon. If we don't have at least n vectors, we can't span $\mathbb{R}^n$.

Theorem 5.16. If $m < n$ then $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$ do not span $\mathbb{R}^n$.

Proof: Row reduce to get to a row of all zeros. Keep track of the operations. Augment, setting the all zero row equal to 1. Reverse the operations. The new augmented vector is not in the span. Here's a worked out example:

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 7 & 0 \\ -1 & 4 & 2 \\ 2 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

▶ Forward Operations:

$$\begin{aligned} 2R_1 + R_2 &\Rightarrow R_2 \\ R_1 + R_3 &\Rightarrow R_3 \\ -2R_1 + R_4 &\Rightarrow R_4 \\ -R_2 + R_3 &\Rightarrow R_3 \\ R_3 &\Leftrightarrow R_4 \end{aligned}$$

pend a column to the right side of the echelon matrix,

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

trix as an augmented matrix, we see that the bottom row is equivalent to the
at the associated linear system has no solutions. If we now reverse the row
iously (as shown in the margin), the first three columns of the augmented
ed to their original form. This gives us

▶ Reverse Operations:

$$\begin{aligned} R_3 &\Leftrightarrow R_4 \\ R_2 + R_3 &\Rightarrow R_3 \\ 2R_1 + R_4 &\Rightarrow R_4 \\ -R_1 + R_3 &\Rightarrow R_3 \\ -2R_1 + R_2 &\Rightarrow R_2 \end{aligned}$$

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 7 & 0 & 0 \\ -1 & 4 & 2 & 1 \\ 2 & -6 & 5 & 0 \end{bmatrix} \tag{6}$$

□

Note that this DOES NOT say that if we have at least n vectors, they have to span $\mathbb{R}^n$. They might, but they might not.

Some things to think about:

1. What is the span of a single vector? Imagine $\vec{v}$ is any nonzero vector in $\mathbb{R}^2$. Draw $\vec{v}$. Draw span $\vec{v}$.

2. If $\vec{u}$ is a multiple of $\vec{v}$, what is span$\{\vec{v}, \vec{u}\}$?

3. If $\vec{u}$ is not a multiple of $\vec{v}$, what is span$\{\vec{v}, \vec{u}\}$?

4. Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ be vectors in $\mathbb{R}^n$. Let $A = [\vec{v}_1 \ldots \vec{v}_m$. If the reduced echelon form of A has at least one row of zeros, what can you say about span$\{\vec{v}_1, \ldots, \vec{v}_m\}$?

6 1.4: Matrix equations $Ax = b$

The vector equations we discussed in the previous section can be written in a more compact form.

Definition 6.1. Let $\vec{a}_1, \ldots, \vec{a}_m \in \mathbb{R}^n$ be vectors. Set $A = [\vec{a}_1 \cdots \vec{a}_m]$. Let $\vec{x} = (x_1, \ldots, x_m)$ with $x_i \in \mathbb{R}$. We define

$$A\vec{x} = x_1\vec{a}_1 + \cdots x_m\vec{a}_m.$$ [1]

From this equation, we should see the following very important theorem:

[1] If you already know matrix multiplication, this should match what you already know. If you do not, take this as a definition for now, and we will discuss matrix multiplication more in the next few weeks.

Theorem 6.2. For a given vector $\vec{b} \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is in the span of the columns of A. In particular, if the columns of A span $\mathbb{R}^n$, then the equation always has a solution.

Example 6.3. Kyle pays very close attention to what he eats post workout, and his macronutrient needs vary greatly from week to week. He typically blends a smoothie with yogurt, banana, avocado and tofu. These have macro-nutrient ratios per 100g as

$$\begin{bmatrix} & Y & B & A & T \\ \text{Protein} & 9 & 1 & 2 & 5 \\ \text{Fat} & 5 & 1 & 15 & 3 \\ \text{Carb} & 4 & 22 & 1 & 2 \end{bmatrix}.$$

If A is the nutrition matrix, then $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is the recipe Kyle follows, and $A\vec{x}$ is the resulting smoothie.

Note that A needs to have the same number of columns as $\vec{x}$ has rows.

Example 6.4. Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$. The system

$$A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

is the same as the augmented matrix:

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 7 \end{array}\right).$$

There are a few properties of the matrix product that are important to note, and one important matrix to be aware of.

Definition 6.5. The **identity matrix** I_n is the $n \times n$ matrix with $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, properties:

- $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$
- $A(c\vec{u}) = cA\vec{u}$
- $I\vec{u} = \vec{u}$.