Your Name

PID

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 30 points.
- Each question should take up roughly 10 minutes of your time. Do not spend too much time on each question.
- You are allowed to have one single sided, handwritten note sheet.
- Cheating will result in a zero and be reported to the University.
- **Show all of your work**. You should prove your answers: for example, if a problem asks "Is .... a group?" a yes or no answer is not enough; you must justify why it is true. If you use a main result from class, be sure to clearly state the result.
- If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.
- GOOD LUCK!

Question	Points	Score
1	7	
2	7	
3	6	
4	5	
5	5	
Total:	30	

1. (a) (1 point) Give the definition of an isomorphism of binary structures.

   (b) (3 points) Is the function $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ given by $\phi(A) = \det A$ an isomorphism of the binary structures $(M_2(\mathbb{R}), \cdot)$ and $(\mathbb{R}, \cdot)$?

   (c) (3 points) Is the function $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n) = n + 1$ an isomorphism of the binary structure $(\mathbb{Z}, +)$ with itself?

2. (a) (1 point) Let G be a group. Give the definition of a subgroup of G.

   (b) (3 points) Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \mid a \in \mathbb{Z} \right\}$. Is S a subgroup of $GL_2(\mathbb{R})$?

   (c) (3 points) Let $S = \{2^n \mid n \in \mathbb{Z}\}$. Is S a subgroup of $(\mathbb{Q}, +)$?

3. For this question, you can use the classification of subgroups of finite cyclic subgroups from class.
   (a) (3 points) Find all subgroups of the group $\mathbb{Z}_{10}$ and give all possible generators for each subgroup.
   (b) (3 points) Find all subgroups of the group $\mathbb{Z}_{11}$ and give all possible generators for each subgroup.

4. (a) (1 point) Give the definition of an abelian group.

   (b) (1 point) Give an example of an abelian group.

   (c) (3 points) If H and K are subgroups of an abelian group G, show that

   $$\{hk \mid h \in H, k \in K\}$$

   is a subgroup of G. Be sure to indicate where you are using the abelian hypothesis.

5. (a) (3 points) If G is a cyclic group with a a generator of G, and $\phi : G \to G'$ is an isomorphism with another group G, prove that G' is cyclic and $\phi(a)$ is a generator of G'.

   (b) (2 points) If $\phi : \mathbb{Z}_6 \to \mathbb{Z}_6$ is an isomorphism of $\mathbb{Z}_6$ with itself, what are the possible values of $\phi(5)$?