Your Name

PID

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 30 points.
- Each question should take up roughly 10 minutes of your time. Do not spend too much time on each question.
- You are allowed to have one single sided, handwritten note sheet.
- Cheating will result in a zero and be reported to the University.
- **Show all of your work**. You should prove your answers: for example, if a problem asks "Is .... a group?" a yes or no answer is not enough; you must justify why it is true. If you use a main result from class, be sure to clearly state the result.
- If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.
- GOOD LUCK!

Question	Points	Score
1	7	
2	7	
3	6	
4	5	
5	5	
Total:	30	

1. (a) (1 point) Give the definition of an isomorphism of binary structures.

Solution: Let $(S, \star)$ and $(S', \star')$ be two binary structures. An **isomorphism** is a one-to-one and onto (or injective and surjective, or bijective) function $\phi : S \to S'$ such that $\phi(x \star y) = \phi(x) \star' \phi(y)$ for all $x, y \in S$.

(b) (3 points) Is the function $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ given by $\phi(A) = \det A$ an isomorphism of the binary structures $(M_2(\mathbb{R}), \cdot)$ and $(\mathbb{R}, \cdot)$?

Solution: This is not an isomorphism because it is not one-to-one. The matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ have $\det A = \det B$, but $A \neq B$.

(c) (3 points) Is the function $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n) = n + 1$ an isomorphism of the binary structure $(\mathbb{Z}, +)$ with itself?

Solution: This is not an isomorphism because it does not satisfy the homomorphism property. Given two integers n and m, $\phi(n + m) = n + m + 1 \neq \phi(n) + \phi(m) = n + 1 + m + 1$.

2. (a) (1 point) Let G be a group. Give the definition of a subgroup of G.

Solution: A **subgroup** H of a group G is a subset of G that

- is closed under the binary operation in G, i.e. for any $a, b \in H$, $ab \in H$,
- contains the identity element $e \in G$, i.e. $e \in H$, and
- for any $a \in H$, $a^{-1} \in H$.

(b) (3 points) Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \mid a \in \mathbb{Z} \right\}$. Is S a subgroup of $GL_2(\mathbb{R})$?

Solution: We need to check the subgroup conditions.

- First, we must show that S is closed under multiplication. Given two elements $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in S$ and $B = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \in S$, we must show that $AB \in S$. To do this, we compute $AB = \begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix} \in S$.
- We must show that the identity is in S, which is clear because $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Finally, we must show that if $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in S$, then $A^{-1} \in S$. To do this, we compute that $A^{-1} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} \in S$.

These are all satisfied, so S is a subgroup of $GL_2(\mathbb{R})$.

(c) (3 points) Let $S = \{2^n \mid n \in \mathbb{Z}\}$. Is S a subgroup of $(\mathbb{Q}, +)$?

Solution: This is **not** a subgroup. One solution: the identity of $(\mathbb{Q}, +)$ is 0 and $0 \notin S$. Another solution: this is not closed under the binary operation in $(\mathbb{Q}, +)$ because $2^1 + 2^2 = 2 + 4 = 6$, and $6 \notin S$. Another solution: the inverse of an element $q \in \mathbb{Q}$ is $-q$, therefore the inverse of 2^n is -2^n and $-2^n \notin S$.

3. For this question, you can use the classification of subgroups of finite cyclic subgroups from class.

   (a) (3 points) Find all subgroups of the group $\mathbb{Z}_{10}$ and give all possible generators for each subgroup.

   Solution: In class, we showed that subgroups of the finite cyclic group $\mathbb{Z}_n$ correspond to the divisors of n, and two elements a and b generate the same subgroup if and only if $\gcd(a, n) = \gcd(b, n)$. The divisors of 10 are $1, 2, 5, 10$ and the corresponding subgroups are:

   - $\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ generated by all elements such that $\gcd(a, 10) = 1$, so the generators are $1, 3, 7$, and 9
   - $\langle 2 \rangle = \{0, 2, 4, 6, 8\}$ generated by all elements such that $\gcd(a, 10) = 2$, so the generators are $2, 4, 6, 8$
   - $\langle 5 \rangle = \{0, 5\}$ and 5 is the only generator
   - $\langle 0 \rangle = \{0\}$ and 0 is the only generator

   (b) (3 points) Find all subgroups of the group $\mathbb{Z}_{11}$ and give all possible generators for each subgroup.

   Solution: The divisors of 11 are 1 and 11 (it is prime) and the corresponding subgroups are:

   - $\mathbb{Z}_{11} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ generated by all elements such that $\gcd(a, 11) = 1$, but because 11 is prime, this is true for any nonzero element in $\mathbb{Z}_{11}$, so the generators are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$
   - $\langle 0 \rangle = \{0\}$ and 0 is the only generator

4. (a) (1 point) Give the definition of an abelian group.

Solution: A group G is **abelian** if its binary operation is commutative. (Alternative: a group is abelian if, for all $a, b \in G$, $ab = ba$.)

(b) (1 point) Give an example of an abelian group.

Solution: All cyclic groups are abelian, so $\mathbb{Z}$ or $\mathbb{Z}_n$ are examples. (There are many other examples: $\mathbb{Q}, \mathbb{R}, \mathbb{Q}^\times, \mathbb{R}^\times, \dots$)

(c) (3 points) If H and K are subgroups of an abelian group G, show that

$$\{hk \mid h \in H, k \in K\}$$

is a subgroup of G. Be sure to indicate where you are using the abelian hypothesis.

Solution: Let us call this set HK. We need to check the subgroup conditions.

- First, we must show that HK is closed under the binary operation. This means, given $h_1k_1 \in HK$ and $h_2k_2 \in HK$, we must show that $h_1k_1h_2k_2 \in HK$. Because G is abelian, $h_1k_1h_2k_2 = h_1h_2k_1k_2$, and because H and K are subgroups, $h_1h_2 \in H$ and $k_1k_2 \in K$, therefore $h_1h_2k_1k_2 \in HK$, as desired.
- We must show that the identity e is in HK. Because H and K are subgroups, we know $e \in H$ and $e \in K$. Therefore, $e = ee \in HK$, as desired.
- Finally, we must show that if $hk \in HK$, then $(hk)^{-1} \in HK$. We know $(hk)^{-1} = k^{-1}h^{-1}$ and because G is abelian, $k^{-1}h^{-1} = h^{-1}k^{-1}$. Because H and K are subgroups, $h^{-1} \in H$ and $k^{-1} \in K$, therefore $(hk)^{-1} = h^{-1}k^{-1} \in HK$, as desired.

Therefore, HK is a subgroup of G.

5. (a) (3 points) If G is a cyclic group with a a generator of G, and $\phi : G \to G'$ is an isomorphism with another group G, prove that G' is cyclic and $\phi(a)$ is a generator of G'.

Solution: Because G is cyclic, there is an element $a \in G$ such that $G = \langle a \rangle$. To show G' is cyclic, we must show there is an element $b \in G'$ such that $G' = \langle b \rangle$. We claim that $b = \phi(a)$. By the homomorphism property, for any $n \in \mathbb{Z}$, $\phi(a^n) = \phi(a)^n$. To show that $G' = \langle \phi(a) \rangle$, we must show that any element $z \in G'$ is of the form $z = \phi(a)^r$ for some integer r. But, ϕ is onto, so there is an element $x \in G$ such that $\phi(x) = z$. Because G is cyclic, $x = a^r$ for some $r \in G$. Therefore, $z = \phi(x) = \phi(a^r) = \phi(a)^r$, so G' is cyclic generated by $\phi(a)$.

(b) (2 points) If $\phi : \mathbb{Z}_6 \to \mathbb{Z}_6$ is an isomorphism of $\mathbb{Z}_6$ with itself, what are the possible values of $\phi(5)$?

Solution: Because $\gcd(5, 6) = 1$, 5 is a generator for $\mathbb{Z}_6$. Therefore, by part (a), $\phi(5)$ must also be a generator for $\mathbb{Z}_6$. The only generators of $\mathbb{Z}_6$ are elements such that $\gcd(a, 6) = 1$, so the only generators of $\mathbb{Z}_6$ are 1 and 5. Therefore, $\phi(5)$ must equal 1 or 5.