Section 9: Alternating Groups and Section 10: Cosets

From last time:

Definition 0.1. A cycle of length 2 is a **transposition.**

For example, (12), (46), (35) are all transpositions.

Theorem 0.2. *Any permutation can be written as a product of transpositions.*

In words, this just says that any permutation (any configuration of tiles!) can be gotten by successively swapping tiles.

Proof. Because every permutation is a product of disjoint cycles, we just need to prove this for a single cycle. If $\sigma = (a_1 a_2 \dots a_k)$, then $\sigma = (a_1 a_k)(a_1 a_{n-1}) \dots (a_1 a_3)(a_1 a_2)$. □

Definition 0.3. A permutation is called **even** if it can be expressed as an even number of permutations and **odd** if it can be expressed as an odd number of permutations.

Example 0.4. $\iota = (12)(12)$ so the identity permutation is even. $(123) = (13)(12)$ is even.

Now, let's prove the Theorem we ended with last time:

Theorem 0.5. *If $n \geq 2$, the collection of all even permutations in S_n forms a subgroup of S_n of order $n!/2$ called the **alternating group**, denoted A_n.*

Proof. First, we will show that this is a subgroup. If we multiply two even permutations, it is still even, so this is closed. We also know $\iota = (12)(12)$ is even. Finally, if σ is any permutation written as a product of transpositions, $\sigma = \tau_1 \dots \tau_n$, then $\sigma^{-1} = \tau_n \dots \tau_1$ (because $\tau_i^{-1} = \tau_i$). Therefore, if σ is even, σ^{-1} is even. So, A_n is a subgroup.

To see that A_n has size $n!/2$, let B_n be the collection of all odd permutations. We will show that A_n and B_n have the same size. Let σ be any permutation in A_n, and define a function $\phi : A_n \to B_n$ by $\phi(\sigma) = (12)\sigma$. (We are **not** claiming this is a homomorphism, just a regular function!) This function is one-to-one because, if $\phi(\sigma) = \phi(\tau)$, $(12)\sigma = (12)\tau$, so $\sigma = \tau$. It is also onto because, given any $\rho \in B_n$, $(12)\rho \in A_n$, and $\phi((12)\rho) = (12)(12)\rho = \rho$. Therefore, ϕ is onto and one-to-one so A_n and B_n must have the same number of elements, $n!/2$. □

We'll explore alternating groups momentarily, but we will first introduce a new word.

Definition 0.6. Let H be a subgroup of a group G. The subset $aH = \{ah \mid h \in H\}$ is the **left coset of H containing** a and the subset $Ha = \{ha \mid h \in H\}$ is the **right coset of H containing** a.

Example 0.7. Let $G = \mathbb{Z}_6$ and $H = \{0, 3\}$. What are the left cosets of H?

We are looking for all possible subsets aH where $a \in \mathbb{Z}_6$. We can just enumerate these over possible elements of $\mathbb{Z}_6$:

- $a = 0$ or $a = 3$: $aH = H = \{0, 3\}$ (Fact: if $a \in H$, because H is closed, $aH = H$)
- $a = 1$ or $a = 4$: $aH = \{1, 4\}$
- $a = 2$ or $a = 5$: $aH = \{2, 5\}$

For the rest of the day, we are going to practice this.