Section 13: Homomorphisms

Today, we are going to talk about homomorphisms. Let's recall the definition, and some related things we have already proven!

Definition 0.1. Let G and G' be groups. A **homomorphism** is a function $\phi : G \to G'$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$.

As a reminder, here are some things we have already proven about homomorphisms:

Proposition 0.2. *Let $\phi : G \to G'$ be a homomorphism. Then,*

- *if e is the identity in G and e' is the identity in G', $\phi(e) = e'$, and*
- *if a is any element of G, $\phi(a^{-1}) = \phi(a)^{-1}$.*

We also have defined the image of a function:

Definition 0.3. Let $f : A \to B$ be a function and let H be a subset of A. The **image** of H under f is the set $f[H] = \{f(h) \mid h \in H\}$.

and proven that:

Proposition 0.4. *Let G and G' be a groups and let $\phi : G \to G'$ be a homomorphism. Then, $\phi[G]$ is a subgroup of G'.*

We will define two more pieces of terminology, and then the rest of today will be dedicated to practicing with these definitions.

Definition 0.5. Let $f : A \to B$ be a function and let K be a subset of B. The **inverse image** of K under f is the set $f^{-1}[K] = \{a \in A \mid f(a) \in K\}$.

Definition 0.6. Let G and G' be a groups and let $\phi : G \to G'$ be a homomorphism. The **kernel** of ϕ is the set $\ker \phi = \{x \in G \mid \phi(x) = e'\}$. Note that $\ker \phi = \phi^{-1}[e']$.