Section 14: Quotient Groups

Today, we're going to introduce a fundamental type of group: *quotient groups.* The textbook refers to these as *factor groups.*

Example 0.1. Find the cosets of $5\mathbb{Z}$ in $\mathbb{Z}$.

We know the answer to this: the cosets are

$$5\mathbb{Z} = \{\dots, -5, 0, 5, 10, \dots\}$$

$$1 + 5\mathbb{Z} = \{\dots, -4, 1, 6, 11, \dots\}$$

$$2 + 5\mathbb{Z} = \{\dots, -3, 2, 7, 12, \dots\}$$

$$3 + 5\mathbb{Z} = \{\dots, -2, 3, 8, 13, \dots\}$$

$$4 + 5\mathbb{Z} = \{\dots, -1, 4, 9, 14, \dots\}$$

We can actually think of *the set of cosets* as a group! There are 5 elements (one for each coset) and we can define the binary operation using the binary operation in $\mathbb{Z}$: to add together the cosets $a + 5\mathbb{Z}$ and $b + 5\mathbb{Z}$, we should get the coset $a + b + 5\mathbb{Z}$.

If we look carefully, we see that every element in each coset has the *same* remainder modulo 5. (For example, all of the elements in $1+5\mathbb{Z}$ have a remainder of 1.) This tells us that the binary operation above is well defined because, if we chose *any* element in $a + 5\mathbb{Z}$ and *any* element in $b + 5\mathbb{Z}$, when we add them together, we will get an element in $a + b + 5\mathbb{Z}$.

In fact, this group is actually isomorphic to $\mathbb{Z}_5$ via the isomorphism $\phi : G \to \mathbb{Z}_n$ given by $\phi(a + \mathbb{Z}) = a \mod n$.

This is an example of a quotient group! We will spend the rest of today defining them in general.

We need an important concept from the worksheet on Wednesday.

Definition 0.2. A subgroup H of a group G is called a **normal** subgroup if, for any $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

Proposition 0.3. *If G is abelian, every subgroup of G is normal.*

Proposition 0.4. *If $\phi : G \to G'$ is a homomorphism, then $\ker\phi$ is a normal subgroup of G.*

Proof. Let h be any element of $\ker\phi$ and $g \in G$. We must prove that $ghg^{-1} \in \ker\phi$. Because ϕ is a homomorphism, $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1}$. Because $h \in \ker\phi$, this implies $\phi(ghg^{-1}) = \phi(g)\phi(g)^{-1} = e'$, so $ghg^{-1} \in \ker\phi$. Hence, $\ker\phi$ is normal. □

Example 0.5. Using this proposition is one way to show that A_n is a normal subgroup of S_n! From the worksheet, we know there is a homomorphism $\phi : S_n \to \mathbb{Z}_2$ given by $\phi(\sigma) = 0$ if σ is even, and $\phi(\sigma) = 1$ if σ is odd. By the previous proposition, $\ker\phi$ is normal, but $\ker\phi = \{\sigma \mid \phi(\sigma) = 0\} = A_n$, so A_n is normal.

Proposition 0.6. *A subgroup H is normal if and only if any of the following conditions hold:*

- *for any $g \in G$, for all $h \in H$, $ghg^{-1} \in H$.*
- *for any $g \in G$, $gHg^{-1} = H$, where $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$.*

- *for any $g \in G$, the left and right cosets containing g are the same, i.e. $gH = Hg$.*

Proof. First, we will show that H is normal if and only if $H = gHg^{-1}$.

If H is normal, then by definition gHg^{-1} is contained in H. Given any $h \in H$, because H is normal, $g^{-1}hg = h_1$ is contained in H, and $h = gh_1g^{-1}$, so $h \in gHg^{-1}$, hence $H \subset gHg^{-1}$. Therefore, $H = gHg^{-1}$.

If $gHg^{-1} = H$ for any $g \in G$, this implies that $ghg^{-1} \in H$, so H is normal.

Now, we show that $gHg^{-1} = H$ if and only if $gH = Hg$. If $\{ghg^{-1} \mid h \in H\} = \{h \in H\}$, then multiplying both sides on the right by g, we get that $\{gh \mid h \in H\} = \{hg \mid h \in H\}$, so $gH = Hg$. Similarly,if $gH = Hg$, multiplying on the right by g^{-1}, we get that $gHg^{-1} = H$. $\square$

Now, we finally define quotient groups.

Definition 0.7. Let G be a group and let H be a normal subgroup of G. Then, the set of distinct cosets of H, denoted by G/H, is called the **quotient group** of G by H. The binary operation is defined by $(aH)(bH) = (ab)H$.

We need to make sure the binary operation is well defined. We know from experience that two different elements a and a' can give the same coset $aH = a'H$, so we need to show that no matter what elements we pick to define the cosets aH and bH, we get the same answer for $(ab)H$.

Suppose we want to compute $(aH)(bH)$. Choosing $a \in aH$ and $b \in bH$, the multiplication rule above gives $(ab)H$. If we choose different representatives $a' \in aH$ and $b' \in bH$ (meaning $a'H = aH$ and $b'H = bH$), then $a' = ah_1$ and $b' = bh_2$ for some $h_1, h_2 \in H$. Then, we get the coset $(ah_1bh_2)H$. But, we know $bH = Hb$ because H is normal, so because $h_1b \in Hb$, there must be some h_3 such that $h_1b = bh_3$. Therefore, $ah_1bh_2 = abh_3h_2$. But, because H is a subgroup, $h_3h_2 \in H$, so $abh_3h_2 \in abH$. Therefore, $(ah_1bh_2)H = (ab)H$, so we get the same coset!

Let us summarize and draw some consequences:

- If H is a normal subgroup of G, then the quotient group G/H is the set of cosets[1] of H.
- The binary operation in G/H is given by $(aH)(bH) = (ab)H$.
- The identity element of G/H is the element H.
- If aH is an element of G/H, the inverse is the element $a^{-1}H$.

[1]Because H is normal, it doesn't matter if you look at left or right cosets!