Section 9: Orbits, Cycles, and Alternating Groups

Cycle notation is useful because it allows us to easily see the group operation, products, and the order of a permutation.

Example 0.1. To see the product of two permutations in cycle notation, we just follow the cycles. For example, if $\sigma = (15)(23)$ and $\tau = (124)$, we can compute $\sigma\tau = (15)(23) \cdot (124)$. The first cycle (the furthest right) says $1 \to 2$, and then the middle cycle says $2 \to 3$, so overall, $1 \to 3$. Similarly, $2 \to 4$, $3 \to 2$, $4 \to 1 \to 5$, and $5 \to 1$. So, our final permutation in cycle notation is

$$\sigma\tau = (13245).$$

You try! Compute $(124) \cdot (15)(23)$. Compute $(12453) \cdot (34)$.

A bonus question: if a permutation σ is written in cycle notation, what is the *order* of σ? (Remember, the order of σ is $|\langle\sigma\rangle|$, which is the minimal integer n such that $\sigma^n = \iota$.)

We'll answer the bonus question, but we need some words first.

Definition 0.2. A permutation is called a **cycle** if it consists of exactly one component in cycle notation (one piece in parentheses). For example, (12345) is a cycle. (125) is a cycle. (34) is a cycle.

The **length** of a cycle is the number of elements in the cycle. For example, (12345) has length 5. (125) has length 3. (34) has length 2.

Every permutation can be written as a **product of disjoint cycles**. (Disjoint means no number is repeated.) For example, (125)(34) is a product of two disjoint cycles.

Theorem 0.3. *If $\sigma = \sigma_1\sigma_2$ is a product of disjoint cycles, then $\sigma = \sigma_1\sigma_2 = \sigma_2\sigma_1$.*

Proof. This is clear because the entries of σ_1 and σ_2 are disjoint. □

Theorem 0.4. *The* ***order*** *of a permutation σ (the minimal positive n such that $\sigma^n = \iota$ is the least common multiple of the lengths of the cycles of σ when written in cycle notation.*

Proof. If σ is a single cycle, then the order of σ is n where n is the length of the cycle.

For general σ, write $\sigma = \sigma_1 \dots \sigma_k$, where σ_i is a cycle of length l_i and these cycles are all disjoint (no numbers are repeated). Then, by the previous theorem, $\sigma^n = \sigma_1^n \dots \sigma_k^n$. If $\sigma^n = \iota$, we must have $\sigma_i^n = \iota$ for each i.

If $\sigma_i^n = \iota$, we must have n be a multiple of l_i. Therefore, the least such integer n such that $\sigma^n = \iota$ is $n = \operatorname{lcm}(l_1, \dots, l_k)$. □

Example 0.5. What is $|\langle(125)(34)\rangle|$? Answer: 6.

Now, let's explore some notions from our last worksheet.

Definition 0.6. A cycle of length 2 is a **transposition.**

For example, (12), (46), (35) are all transpositions.

Theorem 0.7. *Any permutation can be written as a product of transpositions.*

In words, this just says that any permutation (any configuration of tiles!) can be gotten by successively swapping tiles.

Proof. Because every permutation is a product of disjoint cycles, we just need to prove this for a single cycle. If $\sigma = (a_1 a_2 \dots a_k)$, then $\sigma = (a_1 a_k)(a_1 a_{n-1}) \dots (a_1 a_3)(a_1 a_2)$. $\square$

The number of transpositions needed to do this is *not* unique, but its parity (even/oddness) is.

Theorem 0.8. *No permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.*

Proof. In order to do this, we will think of permutations as *matrices*. Given a permutation, we can think of it telling us how to rearrange the rows of the $n \times n$ identity matrix. For example, the permutation (123) would say the first row moves to the second row, the second to the third, and the third to the first, so the associated matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Each permutation σ can be thought of as such a matrix A_σ.

A transposition switches two rows of the matrix, which changes the determinant by -1. So, if A_σ is the matrix obtained by applying σ to the rows of I, $\det A_\sigma = 1$ if σ can be written as an even number of transpositions and $\det A_\sigma = -1$ if σ can be written as an odd number of permutations. Because $1 \neq -1$, σ cannot be written as both an even and odd number of permutations. $\square$

Definition 0.9. A permutation is called **even** if it can be expressed as an even number of permutations and **odd** if it can be expressed as an odd number of permutations.

Example 0.10. $\iota = (12)(12)$ so the identity permutation is even. $(123) = (13)(12)$ is even.

Theorem 0.11. *If $n \geq 2$, the collection of all even permutations in S_n forms a subgroup of S_n of order $n!/2$ called the* ***alternating group****, denoted A_n.*

Proof. First, we will show that this is a subgroup. If we multiply two even permutations, it is still even, so this is closed. We also know $\iota = (12)(12)$ is even. Finally, if σ is any permutation written as a product of transpositions, $\sigma = \tau_1 \dots \tau_n$, then $\sigma^{-1} = \tau_n \dots \tau_1$ (because $\tau_i^{-1} = \tau_i$). Therefore, if σ is even, σ^{-1} is even. So, A_n is a subgroup.

To see that A_n has size $n!/2$, let B_n be the collection of all odd permutations. We will show that A_n and B_n have the same size. Let σ be any permutation in A_n, and define a function $\phi : A_n \to B_n$ by $\phi(\sigma) = (12)\sigma$. (We are **not** claiming this is a homomorphism, just a regular function!) This function is one-to-one because, if $\phi(\sigma) = \phi(\tau)$, $(12)\sigma = (12)\tau$, so $\sigma = \tau$. It is also onto because, given any $\rho \in B_n$, $(12)\rho \in A_n$, and $\phi((12)\rho) = (12)(12)\rho = \rho$. Therefore, ϕ is onto and one-to-one so A_n and B_n must have the same number of elements, $n!/2$. $\square$