Final Practice

Exam topics include:

- Binary operations
- Isomorphic binary structures
- (Definition of) groups and basic properties
- Subgroups
- Cyclic groups
- Description of all cyclic groups and subgroups of all cyclic groups
- Permutations
- Symmetric groups
- Alternating groups
- Cayley's Theorem
- Cosets
- Lagrange's Theorem
- Corollaries of Lagrange's Theorem
- Direct products
- Homomorphisms
- Quotient groups
- The Fundamental Homomorphism Theorem
- Simple groups
- Group actions

If you are looking through your notes and are unsure if you need to know something for the exam, please ask. The following pages contain practice problems divided into four parts: definitions, big theorems, explicit computations, and proofs.

The final exam will be 8 - 9 questions. It will be similar in style to the midterms. Problems will look similar to those on this practice exam, although the actual final will be much shorter than this practice. Some problems on the exam may also come from old homework assignments or worksheets.

1. Practice with definitions.
   (a) Give the definition of a group.
   (b) Give the definition of a cyclic group.
   (c) Give the definition of an abelian group.
   (d) Give the definition of a subgroup.
   (e) Let G be a group. Give the definition of the order of G, $|G|$.
   (f) Let G be a group and let $a \in G$. Give the definition of the order of a.
   (g) Let A be a set. Give the definition of a permutation of A.
   (h) Give the definition of the symmetric group.
   (i) Give the definition of the alternating group.
   (j) Give the definition of the dihedral group.
   (k) Let H be a subgroup of a group G. Give the definition of a left coset of H.
   (l) Let G_1 and G_2 be groups. Give the definition of the direct product group $G_1 \times G_2$.
   (m) Give the definition of a normal subgroup.
   (n) Let H be a normal subgroup of a group G. Give the definition of the quotient group G/H.
   (o) Give the definition of a homomorphism from G to G'.
   (p) Give the definition of an isomorphism from G to G'.
   (q) Let $\phi : G \to G'$ be a homomorphism. Give the definition of the kernel of ϕ, $\ker \phi$.
   (r) Let $\phi : G \to G'$ be a homomorphism and let H be a subgroup of G. Give the definition of the image of H, $\phi[H]$.
   (s) Let $\phi : G \to G'$ be a homomorphism and let H' be a subgroup of G'. Give the definition of the inverse image of H', $\phi^{-1}[H']$.
   (t) Give the definition of a simple group.
   (u) Give the definition of a group action of G on a set X.
   (v) Let G be a group acting on a set X and let $x \in X$. Give the definition of the stabilizer of x, G_x.
   (w) Let G be a group acting on a set X and let $x \in X$. Give the definition of the orbit of x, $G \cdot x$.
   (x) Let G be a group acting on a set X and let $g \in G$. Give the definition of the fixed points of g, X_g.

2. Practice with big theorems. These are things you can state and use on the exam without proving them.
   (a) State the classification of subgroups of finite cyclic groups $\mathbb{Z}_n$.
   (b) State Cayley's Theorem.
   (c) State Lagrange's Theorem.
   (d) State the Fundamental Homomorphism Theorem.

3. Practice with explicit groups and properties.

   (a) Is $(\mathbb{Z}, \star)$ a group, where $a \star b = a - b$?

   (b) Is $(\mathbb{Q}, \star)$ a group, where $a \star b = ab/2$?

   (c) Find all subgroups of the group $\mathbb{Z}_{10}$ and give all possible generators for each subgroup.

   (d) Find all subgroups of the group $\mathbb{Z}_{12}$ and give all possible generators for each subgroup.

   (e) Give all possible generators of $M_9 = \{a \in \mathbb{Z}_9 \mid \gcd(a, 9) = 1\}$.

   (f) Let $D = \{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}, ab \neq 0\}$. Is D a subgroup of $GL_2(\mathbb{R})$? Is D a normal subgroup of $GL_2(\mathbb{R})$?

   (g) Let $\sigma = (123)(45) \in S_6$. What is the order of σ?

   (h) Let $\sigma = (123)(45) \in S_6$. What is σ^{-1}?

   (i) Let $\sigma = (123)(45)$ and let $\tau = (235)$. What is $\sigma\tau$?

   (j) What is the order of the element $(2, 5)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{15}$?

   (k) What is the order of the element $(2, 5)$ in $\mathbb{Z}_3 \times \mathbb{Z}_8$?

   (l) Prove that D_5 is not isomorphic to a subgroup of S_4.

   (m) List the left cosets of $H = \langle(12)\rangle$ in S_3. Are the left cosets equal to the right cosets?

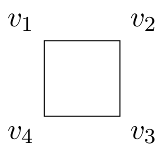
   (n) Show that $\phi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $\phi(n) = (n, 2n)$ is a homomorphism. What is $\ker \phi$? What is $\phi[\mathbb{Z}]$?

   (o) Show that $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ given by $\phi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = a + d$ is a homomorphism. What is $\ker \phi$?

   (p) Let $G = \mathbb{Z}_8$. List the cosets of $H = \{0, 4\}$. Find a group G' and homomorphism $\phi : G \to G'$ such that $\ker \phi = H$ and use it to determine to what group G/H is isomorphic.

   (q) Let $G = \mathbb{Z} \times \mathbb{Z}$ and let $H = \langle(-1, 1)\rangle$. List the cosets of H in G. Find a homomorphism $\phi : G \to \mathbb{Z}$ with $\ker \phi = H$, and use it to determine to what group G/H is isomorphic.

   (r) Let $X = \{v_1, v_2, v_3, v_4\}$ be the vertices of the square:



   D_4 acts on X. What is G_{v_2}?

   (s) Let $X = \mathbb{Z} \times \mathbb{Z}$ and let $G = \mathbb{Z}$. Is the function $G \star X$ given by $a \star (n, m) = (n + a, m - a)$ a group action? What is $G \cdot (1, 1)$?

4. Practice with proofs.

   (a) If G is abelian and $\phi : G \to G'$ is an onto homomorphism, prove that G' is abelian. Give an example to show that this is false if ϕ is not onto.

   (b) If G is a group such that $x^2 = e$ for all $x \in G$, prove that G is abelian.

(c) For sets H and K, define the intersection $H \cap K$ to be

$$H \cap K = \{x \mid x \in H \text{ and } x \in K\}.$$

Show that if H and K are subgroups of a group G, then $H \cap K$ is a subgroup of G. Furthermore, show that if H and K are normal, then $H \cap K$ is normal.

(d) Let G_1 and G_2 be groups and let $G_1 \times G_2$ be the direct product. Prove that, if $|G_1| = p$ and $|G_2| = q$, where p and q are distinct prime numbers, then $G_1 \times G_2$ is cyclic.

(e) Let G_1, G_2 be groups and let $\phi : G_1 \times G_2 \to G_1$ be the function $\phi(g_1, g_2) = g_1$. Prove that ϕ is a homomorphism and prove that $\ker \phi \cong G_2$.

(f) Let G and G' be finite groups and let $\phi : G \to G'$ be a group homomorphism. Show that $|\phi[G]|$ divides $|G|$ and $|G'|$. Using this, determine all homomorphisms $\phi : \mathbb{Z}_r \to \mathbb{Z}_s$ where $\gcd(r, s) = 1$.

(g) Let $\phi : G \to G'$ be any group homomorphism. Prove that $\ker \phi$ is a subgroup of G and that $\ker \phi$ is a normal subgroup of G.

(h) Let $\phi : G \to G'$ be an onto homomorphism. Let N be a normal subgroup of G. Show that $\phi[N]$ is a normal subgroup of G'.

(i) For $n \geq 2$, prove that S_n is not a simple group.