Homework 1: Due Thursday, January 16

1. **Problem 1.** Section 2, Problem 7: Let $\star$ be the binary operation on $\mathbb{Z}$ defined by $a \star b = a - b$. Is $\star$ commutative? Is it associative?

2. **Problem 2.** Section 2, Problem 10: Let $\star$ be the binary operation on $\mathbb{Z}^+$ defined by $a \star b = 2^{ab}$. Is $\star$ commutative? Is it associative?

3. **Problem 3.** Section 2, Problem 23: Let H be the subset of $M_2(\mathbb{R})$ consisting of matrices of the form
$$H = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} | a, b \in \mathbb{R} \right\}.$$
   (a) Is H closed under matrix addition?
   (b) Is H closed under matrix multiplication?

4. **Problem 4.** Section 2, Problem 28: Either prove the statement or give a counterexample. *Every commutative binary operation on a set having just two elements is associative.*

   In problems 5, 6, and 7, determine whether or not ϕ is an isomorphism of binary structures. If it is not an isomorphism, why not?

5. **Problem 5.** Section 3, Problem 2: $(\mathbb{Z}, +)$ with $(\mathbb{Z}, +)$ where $\phi(n) = -n$ for all $n \in \mathbb{Z}$.

6. **Problem 6.** Section 3, Problem 3: $(\mathbb{Z}, +)$ with $(\mathbb{Z}, +)$ where $\phi(n) = 2n$ for all $n \in \mathbb{Z}$.

7. **Problem 7.** Section 3, Problem 8: $(M_2(\mathbb{R}), \cdot)$ with $(\mathbb{R}, \cdot)$ where $\phi(A)$ is the determinant of the matrix A.

8. **Problem 8.** Section 4, Problem 1: Conisder the binary operation $\star$ on $\mathbb{Z}$ defined by $a \star b = ab$. Decide whether $(\mathbb{Z}, \star)$ is a group. If not, which axioms fail?

9. **Problem 9.** Section 4, Problem 10: Let n be a positive integer and let $n\mathbb{Z} = \{nm | m \in \mathbb{Z}\}$.
   (a) Show that $(n\mathbb{Z}, +)$ is a group.
   (b) Show that $(n\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Z}, +)$.

10. **Problem 10.** Section 4, Problem 24: Give a table for a binary operation on the set $\{e, a, b\}$ of three elements satisfying axioms $\mathcal{G}_2$ and $\mathcal{G}_3$ but not axiom $\mathcal{G}_1$.

11. **Problem 11.** Section 4, Problem 29: Show that if G is a group with identity e and with an even number of elements, then there is $a \neq e$ in G such that $a \star a = e$.

12. **Problem 12.** Section 4, Problem 32: Show that every group G with identity e such that $x \star x = e$ for all $x \in G$ is abelian.