Homework 2: Due Tuesday, January 28

1. **Problem 1.** Section 5, Problem 3: Is $7\mathbb{Z}$ a subgroup of $(\mathbb{C}, +)$? If yes, prove it. If not, explain why not.

2. **Problem 2.** Section 5, Problem 10: Is the set $\{\pi^n \mid n \in \mathbb{Z}\}$ a subgroup of $(\mathbb{C}, +)$? If yes, prove it. If not, explain why not.

3. **Problem 3.** Section 5, Problem 12: Is the set $S = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = \pm 1\}$ a subgroup of $(\mathrm{GL}_n(\mathbb{R}), \cdot)$? If yes, prove it. If not, explain why not.

4. **Problem 4.** Section 5, Problem 13: Is the set $S = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^T A = I_n\}$ a subgroup of $(\mathrm{GL}_n(\mathbb{R}), \cdot)$? If yes, prove it. If not, explain why not. (Here, I_n is the $n \times n$ identity matrix, and A^T is the transpose of A.)

5. **Problem 5.** Find the order of the cyclic subgroup generated by the given element.

   (a) Section 5, Problem 27: The subgroup of $(\mathbb{Z}_4, +)$ generated by 3.

   (b) (Not from book): The subgroup of $(\mathrm{GL}_2(\mathbb{R}), \cdot)$ generated by $\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$ (Hint: rotation.)

   (c) Section 6, Problem 20: The subgroup of $(\mathbb{C}^x, \times)$ generated by $(1+i)/\sqrt{2}$ (Hint: where is this point on the circle?)

6. **Problem 6.** Section 5, Problem 43: If H and K are subgroups of an abelian group G, show that
$$\{hk \mid h \in H, k \in K\}$$
is a subgroup of G.

7. **Problem 7.** Section 5, Problem 54: For sets H and K, define the intersection $H \cap K$ to be
$$H \cap K = \{x \mid x \in H \text{ and } x \in K\}.$$
Show that if H and K are subgroups of a group G, then $H \cap K$ is a subgroup of G.

8. **Problem 8.** Find all subgroups of the given group. (You do not need to draw a group diagram.)

   (a) Section 6, Problem 22: $\mathbb{Z}_{12}$

   (b) Section 6, Problem 23: $\mathbb{Z}_{36}$

9. **Problem 9.** Let a and b be elements of a group G. Show that if ab has finite order n, then ba also has order n.

10. **Problem 10.** Section 6, Problem 51: Let p and q be distinct prime numbers. Find the number of generators of the cyclic group $\mathbb{Z}_{pq}$.