Homework 3: Due Thursday, February 13

1. Problem 1. Let \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \) and \( \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} \)
   
   (a) Section 8, Problem 1: Compute \( \tau \sigma \).
   
   (b) Section 8, Problem 2: Compute \( \tau^2 \sigma \).
   
   (c) Section 8, Problem 4: Compute \( \sigma^{-2} \tau \).

2. Problem 2. For \( \sigma \) and \( \tau \) as in Problem 1, compute the following.
   
   (a) Section 8, Problem 6: \( |\langle \sigma \rangle| \).
   
   (b) Section 8, Problem 7: \( |\langle \tau^2 \rangle| \).
   
   (c) Section 8, Problem 8: \( \sigma^{100} \).

3. Problem 3.
   
   (a) Section 8, Problem 16: Find the number of elements in the set \( \{ \sigma \in S_4 \mid \sigma(3) = 3 \} \).
   
   (b) Show that the set from part (a) is a subgroup of \( S_4 \).

4. Problem 4. Section 9, Problem 7. Consider the permutation \( \sigma = (145)(78)(257) \in S_8 \).
   Write \( \sigma \) in cycle notation.

5. Problem 5.
   
   (a) Section 9, Problem 14: Find the maximum possible order for an element of \( S_5 \).
   
   (b) Section 9, Problem 15: Find the maximum possible order for an element of \( S_6 \).

6. Problem 6. Section 9, Problem 23(f) (with slight modification): Show that \( S_n \) is not cyclic for any \( n \geq 3 \).

7. Problem 7. Section 9, Problem 23(g) (with slight modification): Is \( A_3 \) an abelian group? Prove yes or no.

8. Problem 8. Section 9, Problem 29: Show that for every subgroup \( H \) of \( S_n \) for \( n \geq 2 \), either all of the permutations in \( H \) are even or exactly half of them are even.

9. Problem 9. Let \( G \) be group. Define the center of \( G \), denoted \( Z(G) \), to be the following subset:
   \[
   Z(G) = \{ x \in G \mid xy = yx \quad \forall y \in G \}.
   \]
   
   (a) Show that \( Z(G) \) is a subgroup of \( G \).
   
   (b) What is the center of \( D_3 \)?
   
   (c) What is the center of \( D_4 \)?

10. Problem 10. Let \( G \) be a finite group. Prove that there exists a positive integer \( N \) such that, for any \( x \in G \), \( x^N = e \).