Homework 4: Due Thursday, February 20

1. **Problem 1.** Section 10, Problem 1: Find all cosets of the subgroup $4\mathbb{Z}$ of $\mathbb{Z}$.
2. **Problem 2.** Section 10, Problem 4: Find all cosets of the subgroup $\langle 4 \rangle$ of $\mathbb{Z}_{12}$
3. **Problem 3.**
   (a) Find all left cosets of $\langle (1234) \rangle$ in S_4.
   (b) Section 10, Problem 15: Find the index of $\langle (1254)(23) \rangle$ in S_5. (Look carefully at the permutation.)
   (c) Section 10, Problem 16: Find the index of $\langle (1245)(36) \rangle$ in S_6.
4. **Problem 4.** In class, we defined the group $(M_n, \cdot_n)$ where $M_n = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$ and $\cdot_n$ is multiplication modulo n.
   (a) Show that, for $n \geq 2$, $H = \{1, n-1\}$ is a subgroup of M_n.
   (b) Find the cosets of H in M_8.
   (c) Use part (a) to show that, for any $n \geq 3$, M_n always has even order.
5. **Problem 5.** On last week's homework, we defined the center of G as the subgroup

   $$Z(G) = \{x \in G \mid xy = yx \quad \forall y \in G\}.$$

   Show that every left coset of $Z(G)$ is also a right coset of $Z(G)$.
6. **Problem 6.** Section 10, Problem 34: Let G be a group of order pq, where p and q are prime numbers. Show that every proper subgroup of G is cyclic.
7. **Problem 7.** Section 10, Problem 37: Show that a group with at least two elements but with no proper nontrivial subgroups must be finite and of prime order. (Hint: first show the group must be cyclic. Then ask: what do you know about subgroups of cyclic groups?)
8. **Problem 8.** Let G be a group such that $|G| = 6$ and assume that, for every $a \in G$, $\text{ord}(a) < 6$. The goal of this problem is to prove that $G \cong S_3$. There are many ways to do this problem, so feel free to use the hints if you'd like or do it your own way.
   (a) Prove that G must have an element a of order 2. (Hint: look at Problem 11 of the first homework assignment.)
   (b) Prove that G must have an element b of order 3. (Hint: if you assume all non-identity elements have order at most 2, you should get a contradiction.)
   (c) Prove that $ab \neq ba$. (Hint: if $ab = ba$, show that you can find an element of order 6.)
   (d) Prove that $G \cong S_3$. (Hint: one way to do this is to make a table for the binary operation in G, and show that it is the same as the table for S_3.)