Homework 5: Due Thursday, March 5

1. **Problem 1.** Section 11, Problem 2: List the elements of $\mathbb{Z}_3 \times \mathbb{Z}_4$. Find the order of each element. Is this group cyclic?
2. **Problem 2.** Section 11, Problem 15: Find the maximum possible order for some element of $\mathbb{Z}_4 \times \mathbb{Z}_6$.
3. **Problem 3.**
   (a) Section 11, Problem 16: Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic?
   (b) Section 11, Problem 20: Are the groups $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ isomorphic?
4. **Problem 4.** Determine if the given map is a homomorphism.
   (a) Section 13, Problem 2: Let $\phi : \mathbb{R} \to \mathbb{Z}$ be given by $\phi(x) =$ the greatest integer $\leq x$.
   (b) Section 13, Problem 3: Let $\phi : \mathbb{R}^\times \to \mathbb{R}^\times$ be given by $\phi(x) = |x|$.
   (c) Let G be an abelian group and let $\phi : G \to G$ be given by $\phi(g) = g^{-1}$. What if G is not abelian?
5. **Problem 5.**
   (a) Suppose $G = \langle a \rangle$ is a cyclic group. Prove that any homomorphism $\phi : G \to G$ is uniquely determined by the value $\phi(a)$. (Meaning: if you know $\phi(a)$, you can determine the entire function ϕ, and if you have two homomorphisms ϕ and ψ, then $\phi(a) = \psi(a)$ if and only if $\phi = \psi$.)
   (b) Section 13, Problem 25: How many homomorphisms are there from $\mathbb{Z} \to \mathbb{Z}$?
   (c) Section 13, Problem 24: How many onto homomorphisms are there from $\mathbb{Z} \to \mathbb{Z}$?
6. **Problem 6.** Section 13, Problem 47: Let $\phi : G \to G'$ be a homomorphism and suppose that $|G'| = p$, a prime number. Prove that ϕ is either the trivial function $\phi(g) = e'$ or ϕ is one-to-one.
7. **Problem 7.** Section 13, Problem 49: Show that, if G, G' and G'' are groups and $\phi : G \to G'$ and $\psi : G' \to G''$ are homomorphisms, then the composition $\psi \circ \phi : G \to G''$ is a homomorphism.
8. **Problem 8.** Find the order of the given quotient group.
   (a) Section 14, Problem 1: $\mathbb{Z}_6/\langle 3 \rangle$.
   (b) Section 14, Problem 6: $(\mathbb{Z}_{12} \times \mathbb{Z}_{18})/\langle (4, 3) \rangle$.
9. **Problem 9.** Section 14, Problem 24: Show that A_n is a normal subgroup of S_n and compute S_n/A_n. That is, find a known group to which S_n/A_n is isomorphic.
10. **Problem 10.** Section 14, Problem 37:
    (a) Show that all automorphisms of a group G form a group under function composition.
    (b) Show that the inner automorphisms of a group G form a normal subgroup of the group in part (a). (You must show both that they form a subgroup and that the subgroup is normal.)