Homework 6: Due Thursday, March 12

1. **Problem 1.** Section 14, Problem 40: Show that
   (a) The $n \times n$ matrices with determinant 1 form a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.
   (b) The $n \times n$ matrices with determinant ± 1 form a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.
2. **Problem 2.** Using Problem 1, is $\mathrm{GL}_n(\mathbb{R})$ simple?
3. **Problem 3.** Let p, q be distinct prime numbers. Is $\mathbb{Z}_p \times \mathbb{Z}_q$ simple?
4. **Problem 4.** Section 15, Problem 34: Show that if a finite group G contains a nontrivial subgroup H of index 2, then G is not simple. (Hint: show that H is normal.)
5. **Problem 5.** Section 15, Problem 36: Let $\phi : G \to G'$ be a group homomorphism, and let N' be a normal subgroup of G'. Show that $\phi^{-1}[N']$ is a normal subgroup of G.
6. **Problem 6.**
   (a) Recall that the center of a group G is
   $$Z(G) = \{x \in G \mid xy = yx \quad \forall y \in G\}.$$
   Show that $Z(G)$ is a normal subgroup.
   (b) Section 15, Problem 37: Show that if $G/Z(G)$ is cyclic, then G is abelian (and hence $Z(G) = G$).
7. **Problem 7.** Section 15, Problem 38: Using Problem 6(b), show that a non-abelian group G of order pq, where p and q are distinct primes, has $Z(G) = \{e\}$.
8. **Problem 8.** Let $X = \{d_1, d_2\}$ be the two diagonals of the square: d_1 connecting the upper left corner to the lower right, and d_2 connecting the upper right to the lower left. Let $G = D_4$. What is G_{d_1}?
9. **Problem 9.** Let X be a set and let G be a group acting on X. Prove that, for any $x \in X$, G_x is a subgroup of G.
10. **Problem 10.** Let $X = \mathbb{R}^2$. Let $G = \mathbb{Z}_n$. Let G act on X by rotating the points of $\mathbb{R}^2$ by an angle of $2\pi a/n$ about the origin, so $a \star (x, y) =$ the rotation of (x, y) by $2\pi a/n$ radians. Describe geometrically $G \cdot (1, 0)$, the orbit of the point $(1, 0)$. For what values of n is $G \cdot (1, 0) = G \cdot (-1, 0)$?