Solutions: Homework 6

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Problem 1. Show that
(a) The $n \times n$ matrices with determinant 1 form a normal subgroup of $GL_n(\mathbb{R})$.
(b) The $n \times n$ matrices with determinant ± 1 form a normal subgroup of $GL_n(\mathbb{R})$.

Proof. (a) Let $H = \{A \in GL_n(\mathbb{R}) | \det A = 1\}$. We know that it is a subgroup of $GL_2(\mathbb{R})$. We have to prove that it is normal. Let $A \in H$ and $B \in GL_2(\mathbb{R})$. Then $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \frac{\det(B)\det(A)}{\det(B)} = \det(A) = 1$. Hence, $BAB^{-1} \in H$ for all $A \in H$ and all $B \in GL_2(\mathbb{R})$. Hence H is a normal subgroup of $GL_2(\mathbb{R})$.
(b) Let $H = \{A \in GL_n(\mathbb{R}) | \det A = \pm 1\}$. We know that it is a subgroup of $GL_2(\mathbb{R})$. We have to prove that it is normal. Let $A \in H$ and $B \in GL_2(\mathbb{R})$. Then $\det(BAB^{-1}) = \det(B)\det(A)\det(B^{-1}) = \frac{\det(B)\det(A)}{\det(B)} = \det(A) = \pm 1$. Hence, $BAB^{-1} \in H$ for all $A \in H$ and all $B \in GL_2(\mathbb{R})$. Hence H is a normal subgroup of $GL_2(\mathbb{R})$. □

Problem 2. Using Problem 1, is $GL_n(\mathbb{R})$ simple?

Proof. $GL_n(\mathbb{R})$ is not simple as the subgroup $H = \{A \in GL_n(\mathbb{R}) | \det A = \pm 1\}$, by Problem 1 above is normal in $GL_n(\mathbb{R})$. Note that $H \neq \{I_n\}$ as the $n \times n$ diagonal matrix with one entry -1 and all the other entries are 1 is in H, but not equal to I. Note that $H \neq GL_n(\mathbb{R})$ because the matrix $2I_n$ has determinant 2^n and so $2I_n \in GL_n(\mathbb{R})$, but $2I_n \notin H$. So, H is proper. □

Problem 3. Let p, q be distinct prime numbers. Is $\mathbb{Z}_p \times \mathbb{Z}_q$ simple?

Proof. Since $\mathbb{Z}_p \times \mathbb{Z}_q$ is abelian, any subgroup would be normal. So if we find a proper subgroup, it has to be normal, and hence $\mathbb{Z}_p \times \mathbb{Z}_q$ cannot be simple. Note that the order of $(1, 0)$ is p. So the subgroup $\langle(1, 0)\rangle$ has order p, and so is a proper subgroup. □

Problem 4. Show that if a finite group G contains a nontrivial subgroup H of index 2, then G is not simple.

Proof. Note that if $g \in H$, then $gH = H = Hg$. So, if we prove that for all $g \notin H, gH = Hg$, that shows that H is a normal subgroup of G, and hence G is not simple. (unless $|G| = 2$) We first show that $gH = G \setminus H$. We know that since H is of index 2, G has two cosets in H, given by H and gH. We know that $H \cup gH = G$ and $H \cap gH = \emptyset$. So, $gH = G \setminus H$. Similarly, we know that the right cosets are H and Hg, and by the same argument, we get that $Hg = G \setminus H$. So, we have $gH = Hg$, and hence H is normal. So G is not simple. □

Problem 5. Let $\phi : G \to G'$ be a group homomorphism, and let N' be a normal subgroup of G'. Show that $\phi^{-1}[N']$ is a normal subgroup of G.

Proof. Let $N = \phi^{-1}[N']$. We first prove that N is a subgroup of G. Note that $\phi(e) = e' \in N'$ as N' is a subgroup of G'. This implies that $e \in \phi^{-1}[N'] = N$. Now, let $g, h \in N$. Then $\phi(g), \phi(h) \in N'$. Since N' is a subgroup of G', we know that $\phi(gh) = \phi(g)\phi(h) \in N'$, and hence $gh \in \phi^{-1}[N']$. Similarly, if $g \in \phi^{-1}[N'], \phi(g) \in N'$ and hence $\phi(g^{-1}) = (\phi(g))^{-1} \in N'$. This shows that $g^{-1} \in \phi^{-1}[N']$. This shows that $\phi^{-1}[N']$ is a subgroup of G. Now, let $g \in G$ and $h \in \phi^{-1}[N']$. Then $\phi(h) \in N'$. So, $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) = \phi(g)\phi(h)(\phi(g))^{-1} \in N'$ as $\phi(h) \in N'$ and N' is a normal subgroup of G'. This proves that $\phi^{-1}[N']$ is a normal subgroup of G. □

Problem 6. (a) Recall that the center of a group G is

$$Z(G) = \{x \in G | xy = yx \forall y \in G\}.$$

Show that $Z(G)$ is a normal subgroup.
(b) Show that if $G/Z(G)$ is cyclic, then G is abelian (and hence $Z(G) = G$.)

Proof. (a) We know that $Z(G)$ is a subgroup of G. Let $g \in G$ and $x \in Z(G)$. Then $gx = xg$ and hence $gxg^{-1} = x \in Z(G)$. So, $Z(G)$ is a normal subgroup of G.
(b) Suppose $G/Z(G)$ is cyclic. Then there exists $g \in G$ such that $G/Z(G) = \langle gZ(G) \rangle$. Let $a, b \in G$. Then $aZ(G) = g^n Z(G)$ for some $n \in \mathbb{Z}$ and $bZ(G) = g^m Z(G)$. This implies that $a = g^n x$ and $b = g^m y$ for some $x, y \in Z(G)$. Then $ab = g^n x g^m y = g^n g^m xy$ as $x \in Z(G)$. So, $ab = g^{n+m} xy$. Now, $ba = g^m y g^n x = g^m g^n yx = g^{m+n} xy$ as $x \in Z(G)$. This implies that $ab = ba$ for all $a, b \in G$. So, G is abelian. □

Problem 7. Using Problem 6(b), show that a non-abelian group G of order pq, where p and q are distinct primes, has $Z(G) = \{e\}$.

Proof. Since G is non-abelian, we should have that $Z(G) \neq G$. So, by Lagrange's theorem, $|Z(G)| = 1, p$ or q. If $|Z(G)| = p$ or q, then $|G/Z(G)|$ is prime, and hence it has to be cyclic, and hence G has to be abelian, a contradiction. So, $|Z(G)| = 1$, i.e. $Z(G) = \{e\}$. □

Problem 8. Let $X = \{d_1, d_2\}$ be the two diagonals of the square: d_1 connecting the upper left corner to the lower right, and d_2 connecting the upper right to the lower left. Let $G = D_4$. What is G_{d_1}?

Proof. $D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$. Here r is anticlockwise rotation of the square, and f is the flip along the vertical symmetry line of the square. Then, $r \star d_1 = d_2$, so $r \notin G_{d_1}$. $r^2 \star d_1 = r \star (r \star d_1) = r \star d_2 = d_1$. So $r^2 \in G_{d_1}$. Now, since G_{d_1} is a subgroup, if $r^3 \in G_{d_1}$, then $r = (r^3)^3$ should also be in G_{d_1}, which is false. Hence $r^3 \notin G_{d_1}$. Now, note that $|G_{d_1}|$ divides $|D_4| = 8$, and $G_{d_1} \neq D_4$. So, $|G_{d_1}| = 4$. Note that $f \star d_1 = d_2$, hence $f \notin G_{d_1}$. But $(fr) \star d_1 = f \star (r \star d_1) = f \star d_2 = d_1$. So, $fr \in G_{d_1}$. Since $fr, r^2 \in G_{d_1}$, we must have $fr^3 \in G_{d_1}$. So, $\{e, r^2, fr, fr^3\} \subset G_{d_1}$. But since $|G_{d_1}| = 4$, we have $G_{d_1} = \{e, r^2, fr, fr^3\}$. □

Problem 9. Let X be a set and let G be a group acting on X. Prove that, for any $x \in X, G_x$ is a subgroup of G.

Proof. $G_x = \{g \in G | g \star x = x\}$. By definition, $e \star x = x$. So, $e \in G_x$. Let $g, h \in G_x$. Then $g \star x = x$ and $h \star x = x$. So, $(gh) \star x = g \star (h \star x) = g \star x = x$. So, $gh \in G_x$. Let $g \in G_x$. Then $g \star x = x$. So, $g^{-1} \star x = g^{-1} \star (g \star x) = (g^{-1}g) \star x = e \star x = x$. So, $g^{-1} \in G_x$. So, G_x is a subgroup of G. □

Problem 10. Let $X = \mathbb{R}^2$. Let $G = \mathbb{Z}_n$. Let G act on X by rotating the points of $\mathbb{R}^2$ by an angle of $2\pi a/n$ about the origin, so $a \star (x, y) =$ the rotation of (x, y) by $2\pi a/n$ radians. Describe geometrically $G.(1, 0)$, the orbit of the point $(1, 0)$. For what values of n is $G.(1, 0) = G.(-1, 0)$?

Proof. a acts on $(1, 0)$ by rotation by $2\pi a/n$.

$$G.(1, 0) = \{a.(1, 0)|a \in \mathbb{Z}_n\} = \{(\cos(2\pi a/n), \sin(2\pi a/n)) : 0 \leq a \leq n - 1\}.$$

So $G.(1, 0)$ can be seen as the points on the unit circle with angles a multiple of $2\pi/n$. We show that $G.(1, 0) = G.(-1, 0)$ iff n is even. Firstly, if n is odd, we prove that $(-1, 0) \notin G.(1, 0)$. Suppose not. Then there exists $0 \leq a \leq n-1$ such that $(-1, 0) = a.(1, 0)$. Note that $(-1, 0)$ is just $(1, 0)$ rotated by π radians and $a.(1, 0)$ is $(1, 0)$ rotated by $2\pi a/n$ radians. Since $(-1, 0) = a.(1, 0)$, we must have $\pi = 2\pi a/n$ and hence $n = 2a$, which gives a contradiction as n is odd. So, $(-1, 0) \notin G.(1, 0)$. This implies that, for n odd, $G.(-1, 0) \neq G.(1, 0)$ as $(-1, 0) \in G.(1, 0)$ but not in $G.(1, 0)$. Now, let n be even. Note that $(n/2).(1, 0) = (-1, 0)$ because $(-1, 0)$ is rotation by $2\pi(n/2)/n = \pi$. So, $(-1, 0) \in G.(1, 0)$. Similarly, for any $a \in G.(-1, 0), a.(-1, 0) = (a.((n/2).(1, 0))) = (a + n/2).(1, 0) \in G.(1, 0)$. This implies that $G.(-1, 0) \subseteq G.(1, 0)$. Note that $|G.(-1, 0)| = |G.(1, 0)| = n$, and hence, we have $G.(1, 0) = G.(-1, 0)$. So, $G.(1, 0) = G.(-1, 0)$ iff n is even. □