Section 14: Quotient Groups

We'll start with some definitions/review from last time.

First, a note on homomorphisms (because your homework problems use these words!).

Definition 0.1. Let G be a group. An **endomorphism** of G is a homomorphism $\phi : G \to G$. An **automorphism** of G is an isomorphism $\phi : G \to G$.

Example 0.2. The function $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n) = 5n$ is an *endomorphism.* The function $\phi(n) = -n$ is an *automorphism.*

Proposition 0.3. *Let G be a group and let g be any element of G. Then, the function $i_g : G \to G$ given by $i_g(x) = gxg^{-1}$ is an automorphism. This is called the **inner automorphism of G by** g by your book, but more commonly called **conjugation** by g.*

Proof. We need to show it is a homomorphism, and that it is onto and one-to-one. To see that it is a homomorphism, we compute $i_g(xy) = gxyg^{-1}$, and $i_g(x)i_g(y) = gxg^{-1}gyg^{-1} = gxyg^{-1}$, so $i_g(xy) = i_g(x)i_g(y)$, therefore it is a homomorphism. To see that it is onto, given any element $y \in G$, we must show that there is a solution to $i_g(x) = y$, or $gxg^{-1} = y$. Solving this for x, we get that $x = g^{-1}yg$, so if $x = g^{-1}yg$, then $i_g(x) = y$, hence i_g is onto. Finally, to see that it is one-to-one, if $i_g(x) = i_g(y)$, then $gxg^{-1} = gyg^{-1}$, so $x = y$. Therefore, i_g is an automorphism. □

Example 0.4. Let $G = GL_2(\mathbb{R})$. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, then $i_A(M) = AMA^{-1}$. Compute $i_A\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)$.

We first find $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ so

$$i_A\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}.$$

Now, back to quotient groups.

Definition 0.5. A subgroup H of a group G is called a **normal** subgroup if, for any $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

Definition 0.6. Let G be a group and let H be a normal subgroup of G. Then, the set of distinct cosets of H, denoted by G/H, is called the **quotient group** of G by H. The binary operation is defined by $(aH)(bH) = (ab)H$.

Here are some properties of quotient groups:

- If H is a normal subgroup of G, then the quotient group G/H is the set of cosets[1] of H.
- The binary operation in G/H is given by $(aH)(bH) = (ab)H$.
- The identity element of G/H is the element H.
- If aH is an element of G/H, the inverse is the element $a^{-1}H$.

[1]Because H is normal, it doesn't matter if you look at left or right cosets!

Finally, let us do some examples.

Example 0.7. Find the order of the quotient group $\mathbb{Z}_6/\langle 3\rangle$.

The elements are the cosets of $H = \langle 3\rangle = \{0, 3\}$ in $\mathbb{Z}_6$. These are $H = \{0, 3\}$, $1+H = \{1, 4\}$, and $2 + H = \{2, 5\}$, so the order is 4.

What is $2 + H + 2 + H$? The definition is $(2 + H) + (2 + H) = 4 + H$, which is not obviously in our list, but we know that the coset $4 + H = \{4, 1\}$, so $4 + H = 1 + H$. Therefore, $(2 + H) + (2 + H) = 1 + H$.

What is the order of $2+H$ in $\mathbb{Z}_6/\langle 3\rangle$? We compute that $(2+H)+(2+H)+(2+H) = 0+H = H$.

Example 0.8. Find the order of the quotient group $\mathbb{Z}_2 \times \mathbb{Z}_4/\langle(1, 2)\rangle$.

We list the elements first. The elements of $\mathbb{Z}_2\times\mathbb{Z}_4 = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)\}$. The subgroup $H = \langle(1, 2)\rangle = \{(0, 0), (1, 2)\}$ and the cosets are $H = \{(0, 0), (1, 2)\}$, $(0, 1)+H = \{(0, 1), (1, 3)\}$, $(0, 2) + H = \{(0, 2), (1, 0)\}$, and $(0, 3) + H = \{(0, 3), (1, 1)\}$, so the order is 4.

What is $(0, 2) + H + (0, 3) + H$? Answer: $(0, 1) + H$.

What is the inverse of $(0, 3) + H$? Answer: $(0, 3) + H + (0, 1) + H = H$, so the inverse is $(0, 1) + H$.

More abstractly, we have proven that $\ker\phi$ is a normal subgroup, where $\phi : G \to G'$ is any homomorphism. So, we can talk about the quotient group $G/\ker\phi$. This is an important group!

Proposition 0.9. *Let H be a normal subgroup of G. The function $\phi : G \to G/H$ given by $\phi(a) = aH$ is an onto homomorphism with $\ker\phi = H$.*

Proof. This is a homomorphism because $\phi(ab) = abH = (aH)(bH) = \phi(a)\phi(b)$. It is onto because, given any coset aH, then $\phi(a) = aH$. Finally, because $H = eH$ is the identity in G/H, $\ker\phi = \{a \mid \phi(a) = H\}$, so $\ker\phi = \{a \mid aH = H\}$. But, $aH = H$ if and only if $a \in H$, so $\ker\phi = H$. $\square$