**SECTION 16: GROUP ACTIONS**

**Definition 0.1.** Let $X$ be a set and let $G$ be a group. An **action of $G$ on $X$** is a function $\star : G \times X \to X$ such that

- $e \star x = x$ for all $x \in X$, and
- $(g_1 g_2) \star x = g_1 \star (g_2 \star x)$ for all $g_1, g_2 \in G$ and $x \in X$.

We also say $G$ **acts on $X$**.

In words, a group action moves an element $x \in X$ to a new element $g \star x \in X$. Let's see some examples.

**Example 0.2.** Let $X = \{1, 2, 3, 4\}$ and let $G = S_4$. Then, $G$ acts on $X$ by $\sigma \star x = \sigma(x)$.

For example, if $\sigma = (123) \in S_4$, $\sigma \star 1 = \sigma(1) = 2$, and $\sigma \star 3 = \sigma(3) = 1$.

**Example 0.3.** Let $X = \{v_1, v_2, v_3, v_4\}$ be the four vertices of a square:

\[
\begin{array}{ccc}
& v_2 & \\
v_1 & & v_3 \\
v_4 & & \\
& v_1 \\
\end{array}
\]

Then, $D_4$ acts on $X$ by $g \star v_i = v_j$ where $v_i$ gets moved to $v_j$ by $g$. If $r =$ rotation 90 degrees clockwise and $f =$ flip across the vertical axis, then $r \star v_1 = v_2$, $r \star v_2 = v_3$, $f \star v_1 = v_2$, $f r \star v_1 = v_1$; etc.

**Example 0.4.** Let $X = \mathbb{Z} \times \mathbb{Z}$. Then, $G = \mathbb{Z}$ acts on $X$ in many ways! One example: by $a \star (n, m) = (n + a, m)$.

**Example 0.5.** Let $X = \mathbb{R}^n$. Then, $G = \text{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by $A \star v = Av$. (This is linear algebra!) For example, if $n = 2$, and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A \star \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$.

**Definition 0.6.** If $G$ acts on $X$ and $g \in G$ is a fixed element, the **fixed points of $g$** is the set $X_g = \{x \in X \mid g \star x = x\}$.

**Example 0.7.** In the previous example, if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $X_A = \{x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \}$, so $X_A = \{ \begin{bmatrix} x \\ y \end{bmatrix} \} = \text{span} \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$.

**Example 0.8.** In the square example, what is $X_{fr}$? These are the elements such that $fr \star v_i = v_i$. We can just check directly: $fr \star v_1 = v_1$; $fr \star v_2 = v_4$; $fr \star v_3 = v_3$; and $fr \star v_4 = v_2$, so $X_{fr} = \{v_1, v_3\}$.
Definition 0.9. If \( G \) acts on \( X \) and \( x \in X \) is a fixed element, then the **stabilizer of** \( x \) is the set \( G_x = \{ g \in G \mid g \ast x = x \} \).

Definition 0.10. If \( G \) acts on \( X \) and \( x \in X \) is a fixed element, then the **orbit of** \( x \) is the set \( G \cdot x = \{ y \in X \mid y = g \ast x \text{ for some } g \in G \} \).

The **stabilizer** is a subset of \( G \). The **orbit** is a subset of \( X \).

Example 0.11. In the square example, what is \( G_{v_1} \)? What is \( G \cdot v_1 \)?

The stabilizer is the set of elements such that \( g \ast v_1 = v_1 \). So, \( G_{v_1} = \{ e, fr \} \). The orbit is the set of possible elements that \( v_1 \) can move to, so \( G \cdot v_1 = \{ v_1, v_2, v_3, v_4 \} = X \).

Example 0.12. Let’s go back to \( X = \mathbb{Z} \times \mathbb{Z}, G = \mathbb{Z} \), and \( G \) acts on \( X \) by \( a \ast (n, m) = (n+a, m) \). What is \( G_{(1,1)} \)? What is \( G \cdot (1,1) \)?

\( G_{(1,1)} = \{ a \in \mathbb{Z} \mid a \ast (1,1) = (1,1) \} \). Because \( a \ast (1,1) = (1+a, 1) \), this is only \( a = 0 \), so \( G_{(1,1)} = \{ 0 \} \).

\( G \cdot (1,1) = \{ (n,m) \mid (n,m) = a \ast (1,1) \text{ for some } a \in \mathbb{Z} \} \). Because \( a \ast (1,1) = (1+a, 1) \), and \( a \) can be any integer, \( G \cdot (1,1) = \{ (n,m) \mid m = 1 \} \).

Proposition 0.13. For any \( x \), the stabilizer \( G_x \) is a subgroup of \( G \).

Proof. Homework! \( \square \)

Theorem 0.14 (Orbit-Stabilizer Theorem). If \( G \) is finite, for any \( x \in X \), \( |G \cdot x| |G_x| = |G| \).

Proof. By Lagrange’s Theorem, because \( |G_x| \) is a subgroup, we know \( |G| = |G_x| (G : G_x) \), where \( (G : G_x) \) is the number of cosets of \( G_x \) in \( G \). So, we just need to show that \( |G \cdot x| \) is the number of cosets of \( G_x \). The size of \( G/G_x \).

To do this, we will define a bijection \( \phi : G \cdot x \rightarrow G/G_x \). Any element of \( G \cdot x \) is of the form \( g \ast x \) for some \( g \), so let \( \phi(g \ast x) = gG_x \). To see that this is well-defined and one-to-one, we need to show \( g_1 \ast x = g_2 \ast x \) if and only if \( g_1G = g_2G \). But, \( g_1 \ast x = g_2 \ast x \) if and only if \( g_1^{-1}g_2 \ast x = x \), if and only if \( g_1^{-1}g_2 \in G_x \). This occurs if and only if \( g_2 \in g_1G_x \), which happens if and only if \( g_2G_x = g_1G_x \). Therefore, \( \phi \) is well defined and one-to-one. It is also onto by definition. Hence, it is a bijection, so \( |G \cdot x| = (G : G_x) \). \( \square \)