Section 16: Group Actions

Definition 0.1. Let X be a set and let G be a group. An **action of G on X** is a function $\star : G \times X \to X$ such that

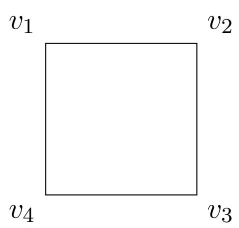
- $e \star x = x$ for all $x \in X$, and
- $(g_1 g_2) \star x = g_1 \star (g_2 \star x)$ for all $g_1, g_2 \in G$ and $x \in X$.

We also say G **acts on** X.

In words, a group action moves an element $x \in X$ to a new element $g \star x \in X$. Let's see some examples.

Example 0.2. Let $X = \{1, 2, 3, 4\}$ and let $G = S_4$. Then, G acts on X by $\sigma \star x = \sigma(x)$.
For example, if $\sigma = (123) \in S_4$, $\sigma \star 1 = \sigma(1) = 2$, and $\sigma \star (3) = \sigma(3) = 1$.

Example 0.3. Let $X = \{v_1, v_2, v_3, v_4\}$ be the four vertices of a square:



Then, D_4 acts on X by $g \star v_i = v_j$ where v_i gets moved to v_j by g. If $r =$ rotation 90 degrees clockwise and $f =$ flip across the vertical axis, then $r \star v_1 = v_2$; $r \star v_2 = v_3$; $f \star v_1 = v_2$; $fr \star v_1 = v_1$; etc.

Example 0.4. Let $X = \mathbb{Z} \times \mathbb{Z}$. Then, $G = \mathbb{Z}$ acts on X in many ways! One example: by $a \star (n, m) = (n + a, m)$.

Example 0.5. Let $X = \mathbb{R}^n$. Then, $G = \mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by $A \star \mathbf{v} = A\mathbf{v}$. (This is linear algebra!) For example, if $n = 2$, and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A \star \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$.

Definition 0.6. If G acts on X and $g \in G$ is a fixed element, the **fixed points of** g is the set $X_g = \{x \in X \mid g \star x = x\}$.

Example 0.7. In the previous example, if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $X_A = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix} \}$, so $X_A = \{ \begin{bmatrix} x \\ x \end{bmatrix} \} = span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Example 0.8. In the square example, what is X_{fr}? These are the elements such that $fr \star v_i = v_i$. We can just check directly: $fr \star v_1 = v_1$; $fr \star v_2 = v_4$; $fr \star v_3 = v_3$; and $fr \star v_4 = v_2$, so $X_{fr} = \{v_1, v_3\}$.

Definition 0.9. If G acts on X and $x \in X$ is a fixed element, then the **stabilizer of** x is the set $G_x = \{g \in G \mid g \star x = x\}$.

Definition 0.10. If G acts on X and $x \in X$ is a fixed element, then the **orbit of** x is the set $G \cdot x = \{y \in X \mid y = g \star x \text{ for some } g \in G\}$.

The *stabilizer* is a subset of G. The *orbit* is a subset of X.

Example 0.11. In the square example, what is G_{v_1}? What is $G \cdot v_1$?
The stabilizer is the set of elements such that $g \star v_1 = v_1$. So, $G_{v_1} = \{e, fr\}$. The orbit is the set of possible elements that v_1 can move to, so $G \cdot v_1 = \{v_1, v_2, v_3, v_4\} = X$.

Example 0.12. Let's go back to $X = \mathbb{Z} \times \mathbb{Z}$, $G = \mathbb{Z}$, and G acts on X by $a \star (n, m) = (n + a, m)$. What is $G_{(1,1)}$? What is $G \cdot (1, 1)$?
$G_{(1,1)} = \{a \in \mathbb{Z} \mid a \star (1, 1) = (1, 1)\}$. Because $a \star (1, 1) = (1 + a, 1)$, this is only $a = 0$, so $G_{(1,1)} = \{0\}$.
$G \cdot (1, 1) = \{(n, m) \mid (n, m) = a \star (1, 1) \text{ for some } a \in \mathbb{Z}\}$. Because $a \star (1, 1) = (1 + a, 1)$, and a can be any integer, $G \cdot (1, 1) = \{(n, m) \mid m = 1\}$.

Proposition 0.13. *For any x, the stabilizer G_x is a subgroup of G.*

Proof. Homework! □

Theorem 0.14 (Orbit-Stabilizer Theorem)**.** *If G is finite, for any $x \in X$, $|G \cdot x||G_x| = |G|$.*

Proof. By Lagrange's Theorem, because $|G_x|$ is a subgroup, we know $|G| = |G_x|(G : G_x)$, where $(G : G_x)$ is the number of cosets of G_x in G. So, we just need to show that $|G \cdot x| =$ the number of cosets of $G_x =$ size of G/G_x.

To do this, we will define a bijection $\phi : G \cdot x \to G/G_x$. Any element of $G \cdot x$ is of the form $g \star x$ for some g, so let $\phi(g \star x) = gG_x$. To see that this is well-defined and one-to-one, we need to show $g_1 \star x = g_2 \star x$ if and only if $g_1 G = g_2 G$. But, $g_1 \star x = g_2 \star x$ if and only if $g_1^{-1} g_2 \star x = x$, if and only if $g_1^{-1} g_2 \in G_x$. This occurs if and only if $g_2 \in g_1 G_x$, which happens if and only if $g_2 G_x = g_1 G_x$. Therefore, ϕ is well defined and one-to-one. It is also onto by definition. Hence, it is a bijection, so $|G \cdot x| = (G : G_x)$. □