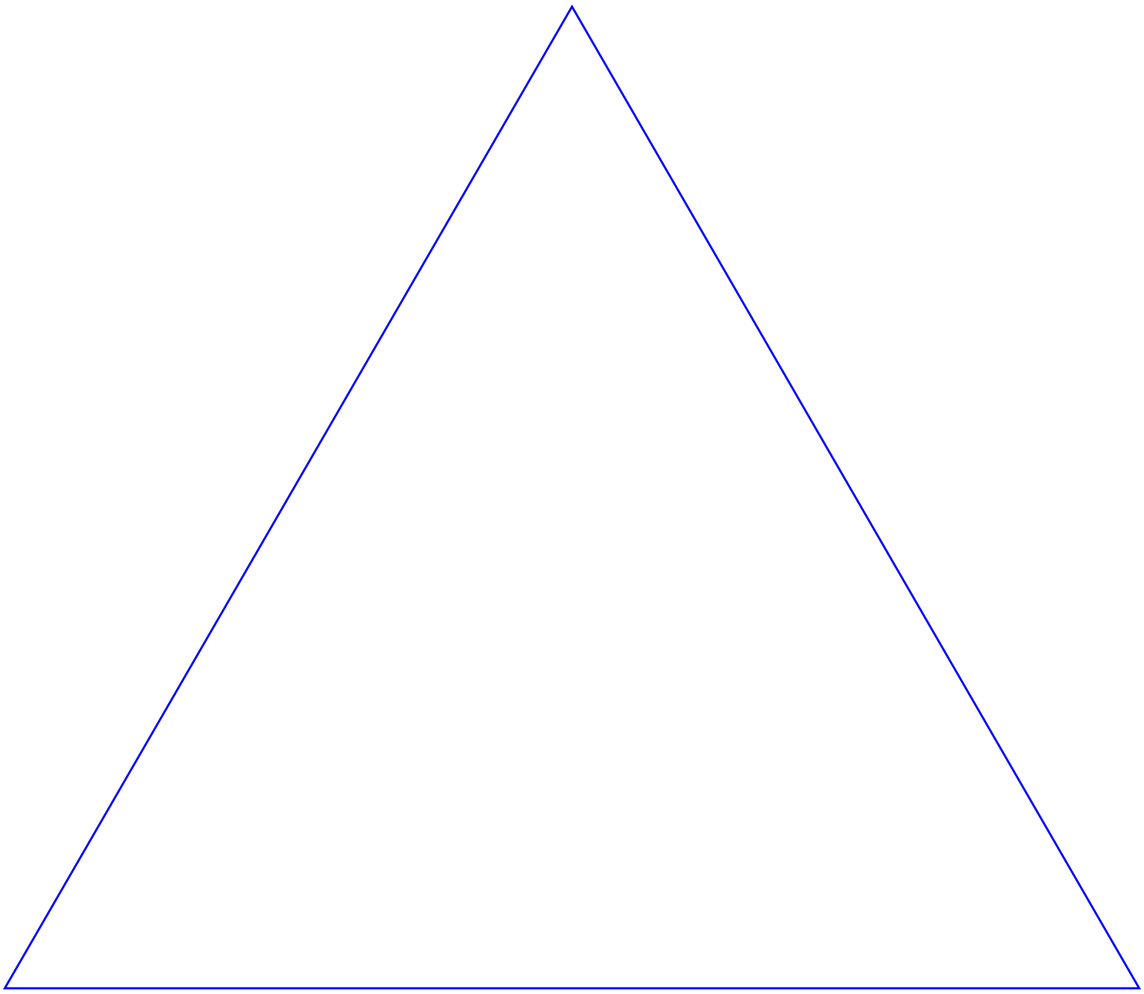
Worksheet 1: Triangles

Welcome to Math 103A! We will dedicate a significant number of classes to working in groups and solving problems, so please take a moment to introduce yourself to your neighbor.

Today, we are going to explore **symmetries** of different shapes. It will be helpful to have a model of one of the simplest shapes. Take a piece of paper and cut out/rip/fold an (approximately) equilateral triangle, large enough for you to move around easily. Don't want to make your own? Look, there's one right here! Feel free to rip along the edges or fold the paper.



Definition 0.1. A symmetry of a figure is a *rigid motion* that maps a figure to itself.

Imagine you have cut the triangle out of this piece of paper. A symmetry is an operation you can perform on the triangle so that it fits exactly back into the hole it was cut from.

(A caveat: for most of today, we will not be using rigorous definitions. The goal is to build intuition for the objects we will be studying in this course. Some of the questions are intentionally vague. Rigor is coming, starting on Wednesday.)

1. How many symmetries does the equilateral triangle have? (Hint: use your triangle and perform rigid motions of it.) Come up with a description of the symmetries. To answer this question, you may first have to determine what makes two symmetries *equivalent.*

2. Prove that you have found all symmetries of the triangle.

3. What happens if you combine two symmetries? Is it a new symmetry or one you have already found?

4. For each symmetry,
   (a) write a description of the symmetry in words,
   (b) draw a diagram to illustrate the symmetry,
   (c) and create a short symbol to represent the symmetry.

5. Let F stand for flipping across the vertical axis and R stand for rotation 120 deg clockwise.
   (a) Show that every operation that you have already found can be written as a combination of (potentially multiple) F's and R's.
   (b) Is this expression unique?

6. Using (4) and (5), determine what happens when do one symmetry operation and then another for every pair of operations. Organize your thoughts in the following table:
   - Leave the top left square blank.
   - Along the top row, list all six symmetry operations using their symbol.
   - Along the left column, list all six symmetry operations using their symbol (in the same order as the top row).
   - In each empty square, fill it in with the symmetry operation you would get by first performing the operation in the left column and then the operation in the top row.
   - After you fill in the table, list at least three observations about it.

Time permitting, here are some more questions.

7. Given any operation T, the **inverse** of T is the operation T^{-1} such that T followed by T^{-1} is the identity (do nothing) operation E. Using the table, find the inverse of each operation.

8. Given any operation T, the **order** of T is the number of times T must be repeated to get the identity (do nothing) operation. Find the order of each symmetry operation.

9. We know each operation is a combination of F and R. Now, let F_1 be a reflection across the vertical axis (the axis meeting the upper vertex), F_2 be reflection across the axis meeting the lower right vertex, and F_3 be reflection across the axis meeting the lower left vertex. Can you write each symmetry operation as a combination of the F_i's?

10. Imagine that the vertices of the triangle are labeled A, B, C. What do the symmetry operations do to the vertices? Show that you can get any relabeling of the vertices using one symmetry operation.

11. How many symmetry operations does a square have? Label the operations and, if you're feeling inspired, make a table for the symmetries of the square like you did for the triangle (or, skip if you wish).

12. If you label the vertices A, B, C, D, can you get all re-labelings of the vertices of the square by performing symmetry operations like we could for the triangle? Prove your answer is correct.

13. How many symmetry operations does a hexagon have? What are the generators?

14. Can you find the symmetries of the triangle inside those of the hexagon? (Can you find an equilateral triangle with the same vertices as three of the hexagon's?) What are the symmetries of the triangle in terms of those for the hexagon?

15. How many symmetry operations does a regular n-gon have? Prove your answer is correct.

16. *Challenge, relating to problem 9.* (We'll learn how to do this later in the course, but it is something to think about for now.) Show that the expression of each symmetry operation in terms of the F_i's is not unique, but the **parity** of the number of F_i's needed to express each operation is.