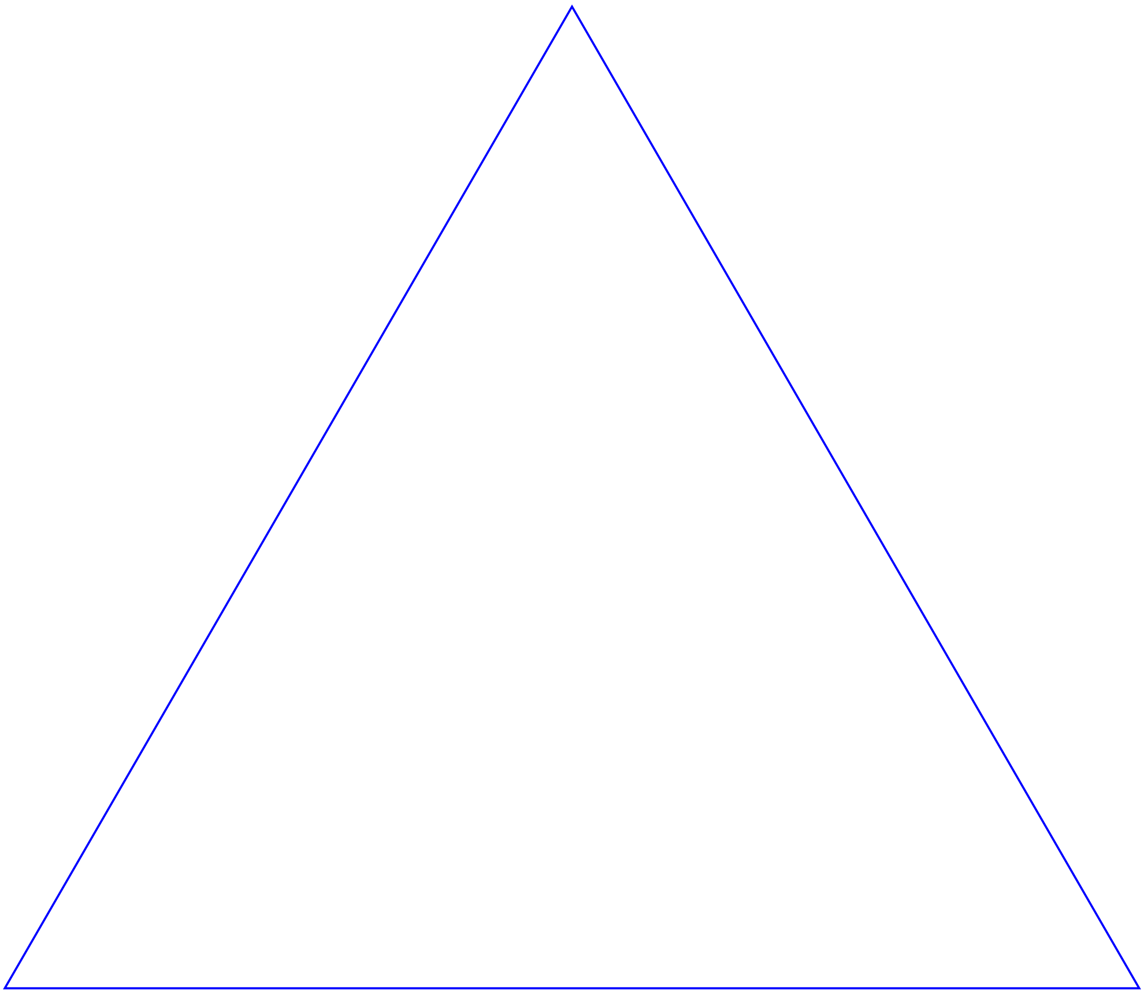
Worksheet 1: Solutions to Selected Problems

Welcome to Math 103A! We will dedicate a significant number of classes to working in groups and solving problems, so please take a moment to introduce yourself to your neighbor.

Today, we are going to explore **symmetries** of different shapes. It will be helpful to have a model of one of the simplest shapes. Take a piece of paper and cut out/rip/fold an (approximately) equilateral triangle, large enough for you to move around easily. Don't want to make your own? Look, there's one right here! Feel free to rip along the edges or fold the paper.



Definition 0.1. A symmetry of a figure is a *rigid motion* that maps a figure to itself.

Imagine you have cut the triangle out of this piece of paper. A symmetry is an operation you can perform on the triangle so that it fits exactly back into the hole it was cut from.

(A caveat: for most of today, we will not be using rigorous definitions. The goal is to build intuition for the objects we will be studying in this course. Some of the questions are intentionally vague. Rigor is coming, starting on Wednesday.)

1. How many symmetries does the equilateral triangle have? (Hint: use your triangle and perform rigid motions of it.) Come up with a description of the symmetries. To answer this question, you may first have to determine what makes two symmetries *equivalent.*

Solution. *There are 6. If F is a flip across the vertical axis and R is a rotation 120 degrees clockwise, the six operations are:*

   (a) *Do nothing, which we denote by E*
   (b) R
   (c) R^2*: this is the same as rotation 120 degrees counterclockwise*
   (d) F
   (e) FR *(convention for now: FR means first do R, then do F, like it would mean if you were composing functions): this is the same as flipping across the axis connecting the lower right vertex with the opposite side*
   (f) $FR^2 = RF$*: this is the same as flipping across the axis connecting the lower left vertex with the opposite side*

2. Prove that you have found all symmetries of the triangle.

Solution. *Here is one proof. Imagine the vertices are labeled A, B, and C. When we perform a symmetry operation, it corresponds to a relabeling of the vertices. There are exactly 6 ways to permute the letters A, B, C, so there can be at most 6 symmetries. Because we have found 6 distinct symmetries, these must be all of them.*

3. What happens if you combine two symmetries? Is it a new symmetry or one you have already found?

Solution. *Because we have found all symmetries of the triangle, it must be one that we have already found.*

4. For each symmetry,
   (a) write a description of the symmetry in words,
   (b) draw a diagram to illustrate the symmetry,
   (c) and create a short symbol to represent the symmetry.

5. Let F stand for flipping across the vertical axis and R stand for rotation 120 deg clockwise.
   (a) Show that every operation that you have already found can be written as a combination of (potentially multiple) F's and R's.
   (b) Is this expression unique?

   Solution. *No: for example, the last operation listed above, FR^2, is the same as RF.*

6. Using (4) and (5), determine what happens when do one symmetry operation and then another for every pair of operations. Organize your thoughts in the following table:
   - Leave the top left square blank.
   - Along the top row, list all six symmetry operations using their symbol.

- Along the left column, list all six symmetry operations using their symbol (in the same order as the top row).
- In each empty square, fill it in with the symmetry operation you would get by first performing the operation in the left column and then the operation in the top row.
- After you fill in the table, list at least three observations about it.

Solution. *REMEMBER: the convention is that FR means* ***first do*** *R and* ***second do*** *F. So, if we want to find the entry in the second column and fourth row, it is "first do F and then do R" which is $RF = FR^2$. The easiest way to fill this in is to first understand/prove that $FR^2 = RF$ and $FR = R^2F$. If you used a different convention or labeling, you will get a different table.*

	E	R	R^2	F	FR	FR^2
E	E	R	R^2	F	FR	FR^2
R	R	R^2	E	FR	FR^2	F
R^2	R^2	E	R	FR^2	F	FR
F	F	FR^2	FR	E	R^2	R
FR	FR	F	FR^2	R	E	R^2
FR^2	FR^2	FR	F	R^2	R	E

Time permitting, here are some more questions.

7. Given any operation T, the **inverse** of T is the operation T^{-1} such that T followed by T^{-1} is the identity (do nothing) operation E. Using the table, find the inverse of each operation.

Solution. *This can be done by finding where E appears in each row. For instance, if we wanted to find the inverse of R, we could go to the second R (corresponding to first doing R) and then scan along the row until we find E, meaning R followed by the action in that column gives E. In this case, this occurs in the third column, so the inverse of R is R^2, as expected.*

8. Given any operation T, the **order** of T is the number of times T must be repeated to get the identity (do nothing) operation. Find the order of each symmetry operation.

Solution. *In this case, we know E has order 1, R and R^2 have order 3, and then F, FR, FR^2 have order 2 (to see they have order 2, we can look in the table: doing one followed by itself gives us E.*

9. We know each operation is a combination of F and R. Now, let F_1 be a reflection across the vertical axis (the axis meeting the upper vertex), F_2 be reflection across the axis meeting the lower right vertex, and F_3 be reflection across the axis meeting the lower left vertex. Can you write each symmetry operation as a combination of the F_i's?

Solution. *Yes. Already in problem 1, we said that $F = F_1$, $FR = F_2$, and $FR^2 = F_3$. Then, we can just play around with the triangle or use the table to see that $R = F_3F_1$ and $R^2 = F_2F_1$.*

10. Imagine that the vertices of the triangle are labeled A, B, C. What do the symmetry operations do to the vertices? Show that you can get any relabeling of the vertices using one symmetry operation.

Solution. *We already mentioned this above, but one way to answer this is just to write out the six relabelings of the vertices and indicate what symmetry operations they correspond to.*

11. How many symmetry operations does a square have? Label the operations and, if you're feeling inspired, make a table for the symmetries of the square like you did for the triangle (or, skip if you wish).

Solution. *There are 8; in the same way as we labeled those of the triangle, we can call F flipping across the vertical axis and R rotation by 90 degrees clockwise. Then, the operations are $E, R, R^2, R^3, F, FR, FR^2, FR^3$ and they satisfy the rules that $R^4 = E$, $F^2 = E$, and $RF = FR^3$.*

12. If you label the vertices A, B, C, D, can you get all re-labelings of the vertices of the square by performing symmetry operations like we could for the triangle? Prove your answer is correct.

Solution. *No! With a rigid motion, we cannot interchange the vertices of the square that are diagonally across from each other. For example, if the vertices are labeled clockwise starting at the upper right so A is diagonally across from C, any rigid motion will keep A diagonally across from C and never next to C. So, we cannot get any symmetry operation that puts A next to C or, similarly, B next to D. Counting the number of labelings that have A across from C, we get 8: we have 4 corners to choose from for A, which determines C, and then have two choices of where to put B, which determines D. These are all relabelings we can get by rigid motions of the square.*

13. How many symmetry operations does a hexagon have? What are the generators?

Solution. *12. Can you prove it? It is generated by F flipping across the vertical axis and R rotating 60 degrees clockwise.*

14. Can you find the symmetries of the triangle inside those of the hexagon? (Can you find an equilateral triangle with the same vertices as three of the hexagon's?) What are the symmetries of the triangle in terms of those for the hexagon?

Solution. *Yes: think of the triangle inside the hexagon gotten by connecting every other vertex. The F for the triangle is the same as the F for the hexagon, but the R for the triangle is R^2 for the hexagon.*

15. How many symmetry operations does a regular n-gon have? Prove your answer is correct.

Solution. *It is always $2n$. Can you prove it?*

16. *Challenge, relating to problem 9.* (We'll learn how to do this later in the course, but it is something to think about for now.) Show that the expression of each symmetry operation in terms of the F_i's is not unique, but the **parity** of the number of F_i's needed to express each operation is.