Worksheet 2: Groups

The best way to understand an abstract definition is to use it. So try it out in the following problems.

1. Which of the following sets are groups with the given binary operation? To show something is a group, you must check that all three axioms are satisfied. To show something is not a group, you must give an example of one axiom failing.
   (a) $(\{1\}, \times)$
   (b) $(\{1, -1\}, \times)$
   (c) $(\mathbb{Z}_n, +_n)$
   (d) $(\mathbb{Q}, \times)$
   (e) $(\mathbb{Q}^\times, \times)$
   (f) $(S = \{a + bi \in \mathbb{C} \mid a^2 + b^2 = 1\}, \cdot)$ (feel free to skip if you haven't seen complex numbers before)
   (g) $(S = \{2^n \mid n \in \mathbb{Z}\}, \times)$
   (h) $(S = \{A \in M_n(\mathbb{R}) | \det A \neq 0\}, \cdot)$
   (i) $(S = \{r \in \mathbb{R} | r \neq -1\}, \star)$ where $a \star b = a + b + ab$
2. We observed that, for the symmetries of the triangle, in each row and column of the table representing the binary operation, each group element appeared exactly once. Prove that this is always true, for any group.
3. Assume that ϕ is an isomorphism between binary structures $(S, \star)$ and $(S', \star')$ such that e is an identity element for S. Prove that $\phi(e)$ is an identity element for S'.
4. Assume that ϕ is an isomorphism between groups $(G_1, \star_1)$ and $(G_2, \star_2)$. Prove that ϕ takes inverses to inverses, i.e. $\phi(a') = \phi(a)'$.

   Here are some harder problems, if you have finished already.
5. For what values of n is $(\mathbb{Z}_n^\times, \cdot_n)$ a group?
6. First, a definition:

   Definition 0.1. The **order** or **size** of a group G, denoted by $|G|$, is the number of elements in G. The **order** of an element $g \in G$ is the minimal positive integer n such that $g^n = e$.

   If n is a prime number, what is the order of all elements of $(\mathbb{Z}_n, +_n)$?
7. Find two non-isomorphic groups of order 4. Prove that they are not isomorphic. (Symmetries could be a good thing to think about!)
8. Fix an integer $n > 0$. Define the group $(C_n, \star)$ to be the set $C_n = \{e, a, a^2, \ldots, a^{n-1}\}$ and define $\star$ to be $a^k \star a^j = a^{k+j}$ subject to the rule $a^n = e$. Prove that this is a group. Is it isomorphic to any of the groups that we have already seen?
9. Is $(C_6, \star)$ isomorphic to $(D_3, \circ)$? (Remember, D_3 was the group of symmetries of the triangle.)