Worksheet 3: Symmetric Groups

Even though I'm gone, I have left you with another worksheet! This is much like the first worksheet where we explored symmetries of the triangle: some of the questions are *intentionally vague* and the goal of this worksheet is to introduce you to a new group that will play an important role in the coming weeks.

To introduce you, we're going to play a 'game' (spoiler: probably not the most fun game you've ever played). Start with 5 tiles numbered $1, 2, 3, 4, 5$ and lay them at random along the squares of a 5×1 rectangle:

We will call the *standard configuration* the most natural one:

1	2	3	4	5

The goal of the game is to move tiles in certain ways and turn any configuration into the standard one.

If you want to try playing, cut/rip out the squares in this standard configuration and move them around on the board at the top of the page (or, use 5 small objects or other pieces of paper representing the numbers 1 - 5):

1	2	3	4	5

SECTION 1: Introduction.

1. How many different configurations of tiles are there?
2. If you are only allowed to swap two tiles at a time, can you always get the tiles into the standard configuration? What is the minimal number of moves needed?
3. If you are only allowed to swap two tiles at a time, but one of them must be the tile in the first position, can you always get the tiles into the standard configuration? If not, how many configurations can be turned into the standard one?
4. If you are only allowed to pick three tiles and cyclically rotate them to the right (so, if you picked the tiles in spots 2, 4, and 5, the tile in 2 would go to 4, 4 would go to 5, and 5 would go to 2), can you always get the tiles into the standard configuration? If not, how many can be turned into the standard one?
5. If you are only allowed to swap the first two tiles or cyclically rotate any three tiles to the right, can you always get the tiles into the standard configuration? If not, how many can be turned into the standard one?
6. If you are only allowed to choose four tiles and swap the contents of the first two, and swap the contents of the second two, can you always get the tiles into the standard configuration? If not, how many can be turned into the standard one?
7. Describe how this game is related to permutations of 5 things. (A *permutation* is a reordering of the numbers 1 through 5.)

SECTION 2: Symmetric Groups.

Definition 0.1. The **symmetric group** S_n is the group of all permutations of n objects. The group has $n!$ elements.

1. I claim that S_n is a group, which means it has a binary operation. In terms of the game from Section 1, a permutation is a reordering of the elements $1, 2, \ldots, n$, so produces another configuration of elements. You may first want to think about how to represent a permutation (maybe the permutation switching the first two tiles is 21345?). Show that *composition* of permutations is a binary operation. In other words, performing one permutation and then another is a binary operation. (Think of 'performing one permutation' as one particular rearrangement of the tiles: something like 'swap the first two tiles' is a permutation.)
2. Show that S_n together with this binary operation is a group. (We know composition is associative. So, you need to check that there is an identity element and that each permutation has an inverse. Given a permutation, how do you find the inverse?)
3. Is S_n cyclic? Is S_n a commutative group? (Try it, using your tiles!)
4. A **transposition** is a permutation that swaps exactly two tiles. How many transpositions are there in S_n?
5. Show that any permutation can be written as a composition of (potentially many) transpositions. How is this related to Problem 2 from Section 1?
6. Thought experiment: how could you find the order of a permutation?