Worksheet 3: Symmetric Groups

SECTION 1: Introduction.

1. How many different configurations of tiles are there?

Solution. There are $5! = 120$ configurations of tiles.

2. If you are only allowed to swap two tiles at a time, can you always get the tiles into the standard configuration? What is the minimal number of moves needed?

Solution. Yes! To get any configuration of tiles, say $a_1a_2a_3a_4a_5$, to the standard one, I could perform this procedure: if $a_1 \neq 1$, switch a_1 with whichever $a_i = 1$ to get a configuration of the form $1b_2b_3b_4b_5$. (I changed letters from a_i to b_i because we do not know which a_i had to be switched with a_1 to make it of this form). Then if $b_2 \neq 2$, switch b_2 with 2 to get $12c_3c_4c_5$, and continue: if $c_3 \neq 3$, switch c_3 with 3 to get $123d_4d_5$, if $d_4 \neq 4$, switch d_4 with d_5 (note: if $d_4 \neq 4$, we must have $d_5 = 4$ and $d_4 = 5$) to get 12345. Therefore, it takes at most four moves to switch any configuration of tiles into the standard one.

3. If you are only allowed to swap two tiles at a time, but one of them must be the tile in the first position, can you always get the tiles into the standard configuration? If not, how many configurations can be turned into the standard one?

Solution. We can always get the tiles into the standard configuration. Let us call a swap of two tiles a **transposition.** The second problem shows that any configuration can be turned into the standard one via a sequence of transpositions. Therefore, it suffices to show that any transposition can be written as a sequence of swaps where one tiles is in the first position. Let a_1 be the tile in the first position. Given any two tiles a_i in the i^{th} spot and a_j in the j^{th} spot, we can achieve the transposition swapping a_i and a_j by first swapping a_i and a_1 so a_i is in position one and a_1 is in position i, and then swapping a_j and a_i to get a_j in position 1 and a_i in position j, and finally swapping a_j with a_1 to get a_j in position i and a_1 back in position one.

4. If you are only allowed to pick three tiles and cyclically rotate them to the right (so, if you picked the tiles in spots 2, 4, and 5, the tile in 2 would go to 4, 4 would go to 5, and 5 would go to 2), can you always get the tiles into the standard configuration? If not, how many can be turned into the standard one?

Solution. We cannot always turn configurations of tiles into the standard one. Here is one proof, although there are many: the configuration 12354 cannot be turned into 12345 by cyclically rotating three tiles. Instead of tiles on a grid, pretend 12345 are the rows of the 5×5 identity matrix. So, 12345 corresponds to

$$A_{12345} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and 12354 is

$$A_{12354} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Because switching two rows of a matrix changes the determinant by -1, $\det A_{12345} = 1$ and $\det A_{12354} = -1$. What happens as we cyclically rotate the three tiles? Well, rotating $a_i \mapsto a_j$, $a_j \mapsto a_k$, and $a_k \mapsto a_i$ can be thought of as first switching a_i and a_j and then switching a_j and a_k. In other words, any rotation of three tiles comes from two swaps (two transpositions). If we apply two transpositions to the rows of the matrix, they each change the determinant by -1, so overall the determinant does not change: $-1 \cdot -1 = 1$. Therefore, A_{12354} cannot be turned into A_{12345} by sequences of these cyclic rotations because they do not change $\det A_{12354}$, and $\det A_{12354} \neq \det A_{12345}$. One can show that exactly 60 configurations of tiles can be turned into the standard one using these cyclic moves.

5. If you are only allowed to swap the first two tiles or cyclically rotate any three tiles to the right, can you always get the tiles into the standard configuration? If not, how many can be turned into the standard one?

Solution. We can show this is possible using the third problem: by that problem, it suffices to show that we can obtain any swap of a given tile with the tile in the first position in this manner. Do a sequence of rotations to get any given tile in the second spot, switch it with the one in the first spot, and then undo the rotations putting everything else back in position.

6. If you are only allowed to choose four tiles and swap the contents of the first two, and swap the contents of the second two, can you always get the tiles into the standard configuration? If not, how many can be turned into the standard one?

Solution. Just like in problem 4, it is not always possible, because two transpositions don't change the determinant of the associated matrix A. So, just as in that problem, A_{12354} cannot be turned into A_{12345}.

7. Describe how this game is related to permutations of 5 things. (A *permutation* is a reordering of the numbers 1 through 5.)

SECTION 2: SYMMETRIC GROUPS.

Definition 0.1. The **symmetric group** S_n is the group of all permutations of n objects. The group has $n!$ elements.

1. I claim that S_n is a group, which means it has a binary operation. In terms of the game from Section 1, a permutation is a reordering of the elements $1, 2, \ldots, n$, so produces another configuration of elements. You may first want to think about how to represent a permutation (maybe the permutation switching the first two tiles is 21345?). Show that *composition* of permutations is a binary operation. In other words, performing one permutation and then another is a binary operation. (Think of 'performing one permutation' as one particular rearrangement of the tiles: something like 'swap the first two tiles' is a permutation.)

Solution. This is a binary operation because it makes sense to do one rearrangement followed by another.

2. Show that S_n together with this binary operation is a group. (We know composition is associative. So, you need to check that there is an identity element and that each permutation has an inverse. Given a permutation, how do you find the inverse?)

Solution. The identity element is the operation *do nothing*, corresponding to not rearranging tiles. The inverse of any permutation is the one undoing what you just did.

3. Is S_n cyclic? Is S_n a commutative group? (Try it, using your tiles!)

Solution. This is not commutative: for example, switching the first two tiles and then rotating the first three tiles cyclically to the right is not the same as doing them in the opposite order. Because it is not commutative, it cannot be cyclic.

4. A **transposition** is a permutation that swaps exactly two tiles. How many transpositions are there in S_n?

Solution. There are $n(n-1)/2$ transpositions.

5. Show that any permutation can be written as a composition of (potentially many) transpositions. How is this related to Problem 2 from Section 1?

Solution. This is exactly problem 2! In that problem, we described an algorithm to write any permutation as a sequence of transpositions.

6. Thought experiment: how could you find the order of a permutation?