Worksheet 4: Cosets

The first page is about S_3; the back is about S_4.

1. Write down all elements of S_3 (cycle notation will be useful for the next problem).

Solution. ι, (12), (13), (23), (123), (132)

2. Write down all elements of A_3 (remember: there should be 3 elements, and they are the permutations in S_3 that are even).

Solution. ι, $(123) = (13)(12)$, $(132) = (12)(13)$

3. Write down all left cosets of A_3. Write down all right cosets of A_3. Are they the same?

Solution. There are two left cosets and two right cosets, and they are the same.

Left cosets:

- $\iota A_3 = (123)A_3 = (132)A_3 = \{\iota, (123), (132)\}$
- $(12)A_3 = (13)A_3 = (23)A_3 = \{(12), (13), (23)\}$

Right cosets:

- $A_3\iota = A_3(123) = A_3(132) = \{\iota, (123), (132)\}$
- $A_3(12) = A_3(13) = A_3(23) = \{(12), (13), (23)\}$

4. (Optional) Want practice with Cayley's Theorem? Cayley's Theorem says $\mathbb{Z}_3$ is isomorphic to a subgroup of S_3. By writing out the isomorphism (remember, it was $\phi(a) = \lambda_a$, where $\lambda_a : \mathbb{Z}_3 \to \mathbb{Z}_3$ is the function $\lambda_a(x) = a + x$) explicitly, find out what subgroup this is. Is this the same subgroup as A_3?

Solution. The elements of $\mathbb{Z}_3$ are $\{0, 1, 2\}$, and they give three permutations: the first is λ_0, which is the function $\lambda_0(x) = 0 + x$, which is the permutation $\lambda_0 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$. Similarly, we can find that $\lambda_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$ and $\lambda_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$. Right now, these don't look exactly like permutations in S_3 because the numbers are $0, 1, 2$ instead of $1, 2, 3$, so we can just relabel them by adding 1 to each number, getting the three permutations $\lambda_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\lambda_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, and $\lambda_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. Finally, writing these in cycle notation, we get three permutations $\iota, (123)$, and (132). This is the same as A_3! In other words, we've just shown that $\mathbb{Z}_3$ is isomorphic to A_3.

5. Write down all elements of S_4 (cycle notation will be useful for the next problem).

Solution. ι, (12), (13), (14), (23), (24), (34), (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23), (1234), (1243), (1324), (1342), (1423), (1432)

6. Write down all elements of A_4.

Solution. ι, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23).

7. Write down all elements of D_4 (remember: if we label the vertices of the square $1, 2, 3, 4$, each symmetry gives a permutation).

Solution. $e = \iota$, $r = (1234)$, $r^2 = (13)(24)$, $r^3 = (1432)$, $f = (12)(34)$, $fr = (24)$, $fr^2 = (14)(23)$, $fr^3 = (13)$

8. Write down all left cosets of D_4. Write down all right cosets of D_4. Are they the same?

Solution. Left cosets:

- If $a = \iota, (13), (24), (12)(34), (13)(24), (14)(23), (1234)$, or (1432), then $aD_4 = D_4 = \{\iota, (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$
- If $a = (12), (132), (124), (34), (1324), (1423), (234)$, or (143), then $aD_4 = \{(12), (132), (124), (34), (1324), (1423), (234), (143)\}$
- If $a = (14), (134), (142), (1243), (1342), (23), (123)$, or (243), then $aD_4 = \{(14), (134), (142), (1243), (1342), (23), (123), (243)\}$

Right cosets:

- $D_4 = \{\iota, (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$
- $D_4(12) = \{(12), (123), (142), (34), (1423), (1324), (134), (243)\}$
- $D_4(14) = \{(14), (143), (124), (1342), (1243), (23), (234), (132)\}$

They are not the same!