Worksheet 5: Homomorphisms

1. For each of the functions below, determine if it is a homomorphism between the given groups. If it is a homomorphism, describe the kernel.

   (a) $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n) = 5n$

   (b) $\phi : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ given by $\phi(A) = \det A$

   (c) $\phi : S_n \to \mathbb{Z}_2$ given by $\phi(\sigma) = 0$ if σ is even and $\phi(\sigma) = 1$ is σ is odd

   (d) $\phi : \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}_3$ given by $\phi(n) = (n \mod 2, n \mod 3)$

   (e) $\phi : \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ given by $\phi(n, m) = m$

2. If $\phi : G \to G'$ is a homomorphism and K is a subgroup of G', prove that $\phi^{-1}[K]$ is a subgroup of G. As a corollary, prove that $\ker \phi$ is a subgroup of G.

3. Let $\phi : G \to G'$ be a homomorphism. Prove that ϕ is one-to-one if and only if $\ker \phi = \{e\}$.

4. A subgroup H of a group G is called a **normal** subgroup if, for any $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

   (a) If G is abelian, prove that every subgroup is normal.

   (b) If $\phi : G \to G'$ is a homomorphism, prove that $\ker \phi$ is a normal subgroup of G.