Worksheet 5: Homomorphisms

1. For each of the functions below, determine if it is a homomorphism between the given groups. If it is a homomorphism, describe the kernel.

   (a) $\phi : \mathbb{Z} \to \mathbb{Z}$ given by $\phi(n) = 5n$

Solution. For any $n, m \in \mathbb{Z}$,

$$\phi(n + m) = 5(n + m) = 5n + 5m = \phi(n) + \phi(m)$$

so this is a homomorphism. The kernel is all n such that $\phi(n) = 0$, or $5n = 0$, which is just $n = 0$.

   (b) $\phi : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ given by $\phi(A) = \det A$

Solution. For any $A, B \in GL_n(\mathbb{R})$,

$$\phi(AB) = \det(AB) = \det(A)\det(B) = \phi(A)\phi(B)$$

so ϕ is a homomorphism. The kernel is all matrices A such that $\phi(A) = 1$, or $\det A = 1$:

$$\ker \phi = \{A \in GL_n(\mathbb{R}) \mid \det A = 1\}.$$

   (c) $\phi : S_n \to \mathbb{Z}_2$ given by $\phi(\sigma) = 0$ if σ is even and $\phi(\sigma) = 1$ is σ is odd

Solution. We need to check that $\phi(\sigma\tau) = \phi(\sigma) + \phi(\tau)$ for any permutations σ, τ. If σ and τ are both even, then $\sigma\tau$ is even, so $\phi(\sigma\tau) = 0 = \phi(\sigma) + \phi(\tau)$. If they are both odd, then $\sigma\tau$ is even, so $\phi(\sigma\tau) = 0 = 1 + 1 = \phi(\sigma) + \phi(\tau)$. If one is even and one is odd, then $\sigma\tau$ is odd, so $\phi(\sigma\tau) = 1$ and $\phi(\sigma) + \phi(\tau) = 0 + 1$ or $1 + 0$, so $\phi(\sigma\tau) = \phi(\sigma) + \phi(\tau)$. Hence, ϕ is a homomorphism. The kernel is A_n.

   (d) $\phi : \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}_3$ given by $\phi(n) = (n \mod 2, n \mod 3)$

Solution. For any $n, m \in \mathbb{Z}$,

$$\phi(n + m) = (n + m \mod 2, n + m \mod 3) = \phi(n) + \phi(m)$$

so ϕ is a homomorphism. The kernel is all integers n such that $n = 0 \mod 2$ and $n = 0 \mod 3$, which happens if and only if n is a multiple of 6. Hence,

$$\ker \phi = 6\mathbb{Z}.$$

   (e) $\phi : \mathbb{Z}_2 \times \mathbb{Z}_3 \to \mathbb{Z}_3$ given by $\phi(n, m) = m$

Solution. For any $(n_1, m_1), (n_2, m_2) \in \mathbb{Z}_2 \times \mathbb{Z}_3$,

$$\phi(n_1 + n_2, m_1 + m_2) = m_1 + m_2 = \phi(n_1, m_1) + \phi(n_2, m_2)$$

so ϕ is a homomorphism. The kernel is all elements (n, m) such that $m = 0$, so

$$\ker \phi = \{(0, 0), (1, 0)\}.$$

2. If $\phi : G \to G'$ is a homomorphism and K is a subgroup of G', prove that $\phi^{-1}[K]$ is a subgroup of G. As a corollary, prove that $\ker \phi$ is a subgroup of G.

Solution. We check the subgroup criteria. Remember, $\phi^{-1}[K] = \{a \in G \mid \phi(a) \in K\}$.

- If $a, b \in \phi^{-1}[K]$, then $\phi(a) \in K$ and $\phi(b) \in K$. Because K is a subgroup of G', it is closed, so $\phi(a)\phi(b) \in K$. Because ϕ is a homomorphism, $\phi(a)\phi(b) = \phi(ab)$, so $\phi(ab) \in K$ and hence $ab \in \phi^{-1}[K]$, so $\phi^{-1}[K]$ is closed.
- Because $\phi(e) = e'$ and $e' \in K$ because K is a subgroup, $e \in \phi^{-1}[K]$.
- If $a \in \phi^{-1}[K]$, $\phi(a) \in K$, and K is a subgroup, so $\phi(a)^{-1} = \phi(a^{-1}) \in K$, so $a^{-1} \in \phi^{-1}[K]$.

Therefore, $\phi^{-1}[K]$ is a subgroup. Because $\ker\phi = \phi^{-1}[e']$ and $\{e'\}$ is a subgroup of G', this implies that $\ker\phi$ is a subgroup.

3. Let $\phi : G \to G'$ be a homomorphism. Prove that ϕ is one-to-one if and only if $\ker\phi = \{e\}$.

Solution. If ϕ is one-to-one, that means if $\phi(a) = \phi(b)$, then $a = b$. We know $\phi(e) = e'$, and if ϕ is one-to-one, for any a such that $\phi(a) = e'$, we must have $e = a$. Hence, the only element such that $\phi(a) = e'$ is $a = e$, so $\ker\phi = \{e\}$.

If $\ker\phi = \{e\}$, we would like to prove ϕ is one-to-one. Let a and b be elements such that $\phi(a) = \phi(b)$. Then, $\phi(a)\phi(b)^{-1} = e'$, and because ϕ is a homomorphism, this implies that $\phi(ab^{-1}) = e'$, so $ab^{-1} \in \ker\phi$. But, $\ker\phi = \{e\}$, so $ab^{-1} = e$, so $a = b$. Hence, ϕ is one-to-one.

4. A subgroup H of a group G is called a **normal** subgroup if, for any $g \in G$ and $h \in H$, $ghg^{-1} \in H$.

(a) If G is abelian, prove that every subgroup is normal.

Solution. If G is abelian, then $ghg^{-1} = gg^{-1}h = h$, so for any $h \in H$, $ghg^{-1} \in G$, hence any subgroup is normal.

(b) If $\phi : G \to G'$ is a homomorphism, prove that $\ker\phi$ is a normal subgroup of G.

Solution. Let h be any element of $\ker\phi$ and $g \in G$. We must prove that $ghg^{-1} \in \ker\phi$. Because ϕ is a homomorphism, $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1}$. Because $h \in \ker\phi$, this implies $\phi(ghg^{-1}) = \phi(g)\phi(g)^{-1} = e'$, so $ghg^{-1} \in \ker\phi$. Hence, $\ker\phi$ is normal.